Since each term f_{n+k} $(k > 0)$ in (1) is the sum of the preceding n terms,

$$f_{n+k} = f_{n+k-1} + f_{n+k-2} + \cdots + f_k, \tag{2}$$

we see that this generalized Fibonacci sequence reduces to the usual Fibonacci sequence when $n = 2$.

Now let $S = \{H^n; TH^n; T^2H^n, HTH^n; T^3H^n, THTH^n, HT^2H^n, H^2TH^n; \ldots\}$ be the sample space for successful experiments in (*). Then f_i is the number of elements in S consisting of exactly i flips. To verify that

$$P(S) = \sum_{i=1}^{\infty} f_i \cdot \frac{1}{2^i}$$

equals 1 and to evaluate

$$e_n = \sum_{i=1}^{\infty} i \cdot f_i \cdot \frac{1}{2^i}, \tag{3}$$

we shall show that

$$B(x) \equiv \sum_{i=1}^{\infty} f_i \cdot x^i = \frac{x^n}{1 - x - x^2 - \cdots - x^n}. \tag{4}$$

Recall that if $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $B(x) = \sum_{k=0}^{\infty} b_k x^k$ over some interval I, the product $C(x) = A(x) \cdot B(x)$ can be obtained by termwise multiplication of the power series for $A(x)$ and $B(x)$. Collecting terms with equal powers of x, we find that the coefficient c_k of x^k in $C(x) = \sum_{k=0}^{\infty} c_k x^k$ $(x \in I)$ is given by

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0. \tag{5}$$

Since $f_{n+k+1} = 2f_{n+k} - f_k$ from (2) and since $f_k > 0$ for $k > n - 1$, we know (the ratio test) that $\sum_{k=0}^{\infty} f_k x^k$ converges to some function $B(x)$ for $|x| \leqslant \frac{1}{2}$. Taking

$$a_k = \begin{cases} 1, & k = 0 \\ -1, & 1 \leqslant k \leqslant n \\ 0, & k > n \end{cases} \quad \text{and} \quad b_k = f_k$$

and using (2), we see that (5) yields $c_n = 1$ and $c_k = 0$ for $k \neq n$. Therefore, $A(x) = 1 - x - x^2 - \cdots - x^n$ and $C(x) = x^n$ yield $B(x)$ as in (4).

Using $B(x)$ and its derivative $B'(x)$, we find immediately that $P(S) = B(\frac{1}{2}) = 1$, and $e_n = \frac{1}{2} B'(\frac{1}{2})$. Since $B'(\frac{1}{2}) = 2^{n+2} - 4$, the answer sought in (*) is $e_n = 2^{n+1} - 2$.

Editor's Note: $B(x)$ is the generating function for the generalized Fibonacci sequence (1). For a further discussion of generating functions in probability theory, see W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1, 3rd ed., Wiley, New York, 1968.

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Queries

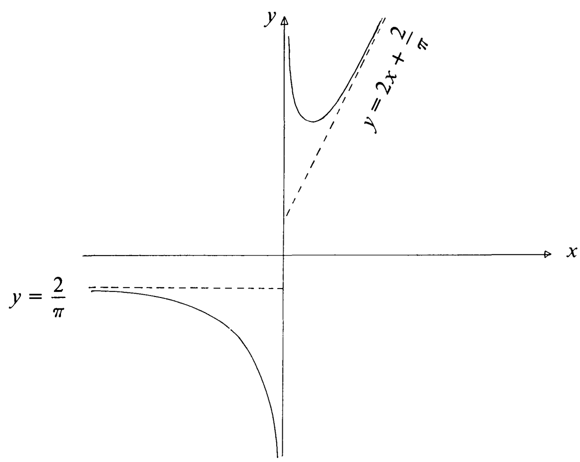
9. Can anyone provide an example of an elementary transcendental function having three different kinds of asymptotes ($x - a = 0, y - b = 0, y = mx + c$) and defined by one single equation in its domain of definition?

[A. Coolsaet, Nazareth, Belgium]

Response. Robert Bumcrot, Hofstra University, Hempstead, NY

The function $y = (1/\arctan x) + x + \sqrt{x^2}$ satisfies Coolsaet's requirements. (For a more general response, consider

$$y = \frac{\pi(c-b)}{4\arctan(x-a)} + \frac{m\left(x+\sqrt{x^2}\right)}{2} + \frac{(b+c)}{2}.)$$



Response. Richard Parris, Phillips Exeter Academy, Exeter, NH

The graph of $y = \coth(x-a)\cdot[\ln(e^{mx+c} + e^b)]$ has $x = a$ as a vertical asymptote. For $m > 0$, we have $y = mx + c$ as an asymptote as $x \to \infty$, and $y = b$ is an asymptote as $x \to -\infty$. For $m < 0$, we have asymptote $y = mx + c$ as $x \to -\infty$ and asymptote $y = b$ as $x \to \infty$.

Response. Gary L. Britton, University of Wisconsin Center, West Bend, WI

The graph of

$$f(x) = (1/x) + (x^2 + x|x|)\sin(1/x)$$

has $y = 0$ as an asymptote as $x \to -\infty$, $x = 0$ as an asymptote as $x \to 0$, and $y = 2x$ as an asymptote as $x \to +\infty$. A generalization which has asymptotes $x - a = 0$, $y - b = 0$, and $y = mx + c$, where $c = b - ma$, is given by

$$F(x) = \frac{1}{x-a} + \frac{m}{2}\left[(x-a)^2 + (x-a)|x-a|\right]\sin\frac{1}{x-a} + b.$$

If it is desired to have each term transcendental, then the first term, $1/(x-a)$, can be replaced by $1/[\exp(x^2 - a^2) - 1]$.