arbitrary partial sum distinct from S_n is of the form $T_{n,j} = S_n + \sum_{i=1}^{j} \frac{1}{k(n+1)-(k-i)}$ for some $j = 1, 2, \ldots, k-1$, and $\left|T_{n,j} - S_n\right| < \frac{1}{n}$.)

2. Prove the geometric-type series identity (2), and verify that the grouping of terms $\left[\left(\sum_{i=1}^{k-1} t^{kn-(k-i)}\right) - (k-1)\, t^{kn}\right]$ can be ignored. (Hint: Take the difference of two geometric series.)
3. Show that for a fixed value of $k \in \{2, 3, \ldots\}$, $\left[\left(\sum_{i=1}^{k-1} \frac{1}{kn-(k-i)}\right) - \frac{k-1}{kn}\right] = \frac{Q(n)}{P(n)}$ where $Q(n)$ is a degree $k-2$ polynomial and $P(n)$ is a degree k polynomial, and establish the convergence of $\sum_{n=1}^{\infty} \left[\left(\sum_{i=1}^{k-1} \frac{1}{kn-(k-i)}\right) - \frac{k-1}{kn}\right]$.

References

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A Game-Like Activity For Learning Cantor's Theorem

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Cantor's theorem, a fundamental result in set theory (and revolutionary in its time), states that a set A cannot be mapped onto its power set $P(A)$. The idea that there exists a hierarchy in the cardinality of infinite sets is a very abstract concept in set theory. Students in mathematics, computer science, and engineering meet it when they take introductory courses in set theory, discrete mathematics, logic, or calculus. Teaching experience indicates that they often find it hard to comprehend. One difficulty is the indirect *proof of nonexistence* which is the heart of Cantor's theorem. We propose here a teaching strategy for explaining this theorem by means of a *constructive game-like activity*, hoping that it would help in communicating this beautiful piece of mathematics to a wide audience of students.

Why is Cantor's theorem difficult to understand?

The fact that an infinite set can be equivalent to a subset of itself is counterintuitive. Nevertheless, students usually find this result rather easy to understand because it is explained by *constructing* one-to- one mappings in various illustrative examples. In fact, classroom experience indicates that the students often develop the (wrong) intuitive speculation that "given two infinite sets, there is a one-to-one correspondence between them, which can be found by looking hard enough" (that is, they tend to guess that all infinite sets are equivalent). Cantor's theorem shows that this is not the case. Let us review a brief version of a proof.

Proof of Cantor's theorem: Suppose that $f : A \to P(A)$ is an *onto* mapping. Define the set D of all elements u of A that do not belong to their match $f(u)$. Since D

is a subset of A, there exists some element y which belongs to A, such that $f(y) = D$. We now arrive at the following contradiction: If y belongs to D, then it *does not* belong to D by the definition of D. For the same reason, if y does not belong to D then it *must* belong to D. This contradiction implies that no *onto* mapping can exist.

Students who first encounter this proof, in the context of infinite sets, often find it cryptic, possibly due to the following difficulties:

1. For finite sets, the result can be illustrated by direct counting: if A contains n elements, then $P(A)$ has 2^n elements. For infinite sets, no "real" examples are provided. Nobody can construct the power set of an infinite set. Actually, nobody can even *list* the elements of such power set! This is one of the *a posteriori* conclusions of the theorem.
2. Proofs of existence are indirect ones, and are therefore more difficult to grasp than proofs that are based on construction. The logic of a *proof of nonexistence* is even more problematic. In our context, it is not enough just to give *one example* of a possible mapping from A to $P(A)$ which is *not onto*, or, for that matter, to give many such examples. The idea is to show that such an onto-mapping *cannot possibly exist*. This can be proved by either exhausting all mapping possibilities (which is of course unfeasible for infinite sets) or via contradiction.

Illustrating Cantor's theorem by a game-like activity

We attempt to bypass some of the above difficulties by illustrating Cantor's theorem by means of a *constructive demonstration.* We call it "the matching game". Through this game-like activity, the students are led to realize how and why the key idea of the proof works for a finite set, typically with a small number of elements, and why it works independently of any counting process. Our classroom experience indicates that this helps them in generalizing the concept to infinite sets.

The matching game: the students are given a game sheet with a set A of M elements and the elements of $P(A)$ (see illustration in Figure 1 left panel). They are requested to "pair" each one of the elements of A with one element of $P(A)$ (by

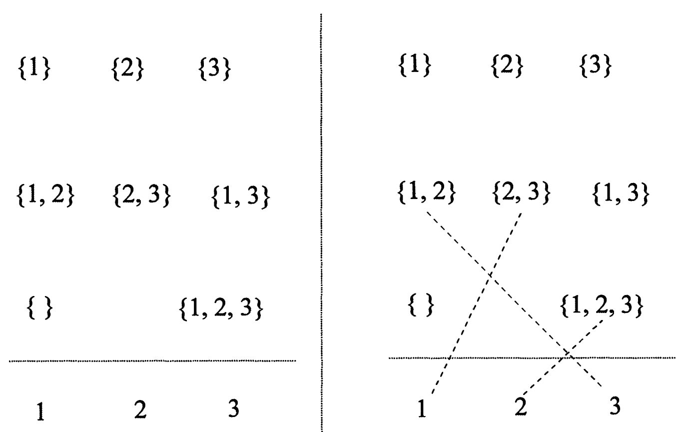

Figure 1. The matching game. Left panel: An empty sheet for $A = \{1, 2, 3\}$. Right panel: a sample attempt (unsuccessful of course) to make an onto-mapping. The "failure" can be "exposed" without looking at the filled up sheet, just by asking the three questions Q1, Q2, Q3 as explained in the text. The subset consisting of the "No" elements is unpaired (though it is not the only one). In this example it is the subset $\{1, 3\}$.

connecting them with a line on the game sheet). After this is done, the instructor identifies a subset of A which remained unmatched, and does this without looking at the student's game sheet. This guess is made after asking the student *only* M yes/no questions, as the following example demonstrates.

Example: An empty game sheet is displayed in Figure 1 (left panel) for the case $M = 3$ and $A = \{1, 2, 3\}$, and a sample matching is demonstrated in the right panel. The matching game, played with this filled up sheet, is the following:
Q1: Is 1 an element of its match? (A1: no)
Q2: Is 2 an element of its match? (A2: yes)
Q1: Is 3 an element of its match? (A3: no)

Without looking at the pairing scheme, the instructor *asserts* that the subset $\{1, 3\}$ is unpaired.

After several successful guesses made by the instructor (e.g., in response to the filled up game sheets of *all* the students in the class) it becomes obvious to the students that the "key" is not likely to be luck (it can be verified that the probability of succeeding randomly in 10 successive rounds, in the discussed example, is less than 1%). Further, the guessed unpaired set changes from one game sheet to the other. This indicates that some matching-specific information per matching is required, and it is found entirely in the three answers A1, A2, A3. As experience shows, the students eventually realize *how* the successful guess is made, namely: the unpaired set is the set of all of the elements that got "No" answers for Q1, Q2, Q3. Now, in addition to *seeing* that this method indeed works, it remains to understand the reason *why* it does. An explanation that was actually made by a student who participated in a demonstration of this activity was: *The set of "No" elements cannot be paired with a "Yes" element since it contains only "No" elements. On the other hand, the set of "No" elements cannot be paired with a "No" element because this element would immediately become a "Yes" element. So, it cannot be paired at all.*

Finally, by emphasizing the fact that the guessing procedure *does not require any counting*, the extension to infinite sets can be understood straightforwardly (although, for the more "rigor oriented" students, it is not obvious how exactly this can be done, as we briefly mention below).

Comparison with the diagonal method

Some textbooks and instructors use the well known diagonal method that proves that the set N (of positive integers) cannot be mapped *onto* the interval [0, 1], as an equivalent to Cantor's theorem. They use the following construction for the proof: attempt to make a list of *all* the numbers in [0, 1], one below the other, so they generate an infinite "array of digits". Then, construct a new number whose i-th digit differs from the i-th diagonal digit in that array. This new number is not on that list, and this generates the desired contradiction. This demonstration and the one proposed here are different. To explain this variant of the diagonal method, one needs first to define the set of real numbers, as well as the *unique* representation of the numbers in [0, 1] as an *infinite* decimal fraction. Doing this rigorously requires a lengthy introduction. Furthermore, using this proof alone might give the students the wrong impression that the hierarchy in infinite sets is somehow related to the algebraic properties of the integers and the real numbers. Finally, this diagonal method cannot be extended beyond the cardinality of N (because, to begin with, one cannot *list* the elements in a row).

These limitations of the "standard" diagonal method were discussed in a presentation of Leron and Moran (1988). They proposed a generalized diagonal method and a game-like activity for illustrating it. Their approach represents an alternative to ours.

Some problems that are still left for thought

As mentioned above, classroom experience indicates that after the idea of constructing the set of "Yes/No" elements is illustrated for small finite sets—its extension to infinite sets is easily understood. However, the gap between intuitive understanding and rigorous treatment can naturally lead to an interesting discussion of the axiomatic basis of set theory. The straightforward "extension" of the game to infinite sets implies that the desired set of "No" elements is identified by considering the answers for all of the questions of the type A1, A2, etc. Is it obvious that this procedure can indeed *be performed* for infinite sets? Does it really have to? And even if the procedure can be completed for countable sets, how would it be extended for an *uncountable* set? To overcome these difficulties, one needs to postulate the existence of the set of "No" elements by merely stating the rule that defines it, and *independently* of any procedure for actually constructing it. This clarifies the need for an axiomatic basis to rely upon.

To conclude, here is a related and amusing puzzle:

Trying to repeat verbatim the proof of Cantor's theorem (for example using the matching game) to show that A cannot be mapped onto $P(A) - \{\phi\}$ does not work. Can you explain why?

Note that the latter statement is clearly correct if A has more than one element, and can be shown by using simple mathematical arguments that reduce it to the standard Cantor theorem. Can you find a version of the matching game that would prove it directly?

Answer:

The original matching game may lead to constructing the empty set which is not in $P(A) - \{\phi\}$. Therefore it cannot work directly as a proof. However, asking two additional questions resolve the situation. This is how: choose any two elements x, y, in A (we assume that A has more than one element). Add the following two questions: "does x belong to the match of y" and "does y belong to the match of x". Considering the responses to these two questions and all the others, define a subset D of A as in the original game. If D turns out to be empty, which means that all elements of A belong to their matches, then we change the choice D according to the following table

Does x belong to the match of y?	Does y belong to the match of x?	D
Yes	Yes	$\{x\}$ (or $\{y\}$)
Yes	No	$\{y\}$
No	Yes	$\{x\}$
No	No	$\{x, y\}$

Reference

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