For example,

$$A^3_{3n} = H_{3n} - H_n = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots + \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{2}{3n},$$

the $(3n)$th partial sum of the series

$$A^3 = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots.$$

To show that A^3 converges, we group the positive pairs and decompose the negative terms, as follows:

$$A^3 = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5}\right) - \cdots,$$

displaying A^3 as an alternating series whose terms decrease in absolute value to zero. The series A^3 therefore converges to a limit. Consequently, (A^3_{3n}), a subsequence of its sequence of partial sums, converges to the same limit. Arguing as in the original case, we find that

$$A^3_{3n} \to \ln 3 \text{ and, therefore, } A^3 \to \ln 3.$$

To see the pattern more fully, note that

$$A^4_{4n} = H_{4n} - H_n = 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \cdots.$$

Let A^k denote the series whose (kn)th partial sum is A^k_{kn}. The proof that $A^k \to \ln k$ goes the same way as the one just given for $k = 3$. (These extensions were noted in a successor manuscript to [**2**], which however died on the editor's desk.)

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1. Frank Burk, Natural logarithms via long division, *College Mathematics Journal* **30** (1999), 309–311.
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Using Differential Equations to Describe Conic Sections

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"A *parabola* is the set of points in a plane that are equidistant from a fixed point F (called the *focus*) and a fixed line (called the *directrix*)." This definition is included in many calculus texts that have a chapter on analytic geometry. Using basic algebra

it is easy to show that the parabola with the focus $F(p, 0)$ and the directrix $x = -p$ has equation $y^2 = 4px$. In some texts, students are asked to show that a ray of light emitted from the focus of a parabola is reflected in the direction perpendicular to the directrix. Using elementary differential equations, we show that this reflection property alone characterizes the parabola. See [**2**, p. 714] for a different proof that uses polar coordinates and differential equations.

Let C be a smooth curve in the plane such that any light ray emitted from F that strikes C is reflected parallel to the x-axis. Assuming that the angle of incidence is equal to that of reflection, we find a differential equation that governs the shape of the curve C. (See Figure 1.)

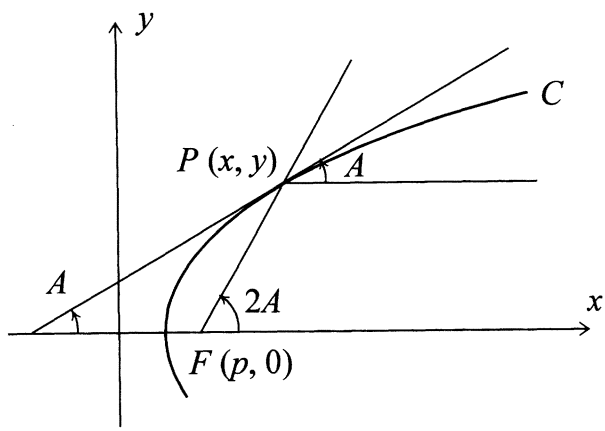

Figure 1.

Suppose $\frac{dy}{dx} = \tan A$. Then $\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{y}{x-p}$, and substituting $\frac{dy}{dx}$ for $\tan A$ we obtain

$$y\left(\frac{dy}{dx}\right)^2 + 2(x-p)\frac{dy}{dx} - y = 0. \tag{1}$$

To find an equation for the curve C, we solve (1) for $\frac{dy}{dx}$ to get the two equations

$$\frac{dy}{dx} = \frac{-(x-p) \pm \sqrt{(x-p)^2 + y^2}}{y}. \tag{2}$$

Since the right-hand expressions are homogeneous functions of $\frac{y}{x-p}$ we use the substitution $u(x) = \frac{y}{x-p}$ to transform (2) to separable equations. Thus,

$$u + (x-p)\frac{du}{dx} = \frac{-1 \pm \sqrt{1+u^2}}{u}, \tag{3}$$

and separating variables yields

$$\int \frac{u}{-u^2 - 1 \pm \sqrt{1+u^2}}\, du = \int \frac{1}{x-p}\, dx.$$

Observe that

$$\int \frac{u}{-u^2 - 1 \pm \sqrt{1+u^2}}\,du = -\int \frac{u(1+u^2)^{-1/2}}{\sqrt{1+u^2} \mp 1}\,du = -\ln\left|\sqrt{1+u^2} \mp 1\right| + C.$$

Therefore, the equations (3) have respective solutions

$$u^2 = \frac{k^2}{(x-p)^2} \pm \frac{2k}{x-p} \quad (k = e^C),$$

and the corresponding solutions of (2) are

$$y^2 = \pm 2k\left(x - p \pm \frac{k}{2}\right). \tag{4}$$

Note that for each real number k, the equations (4) describe a pair of parabolas with vertices $V_1\,(p - \frac{k}{2}, 0)$ and $V_2\,(p + \frac{k}{2}, 0)$.

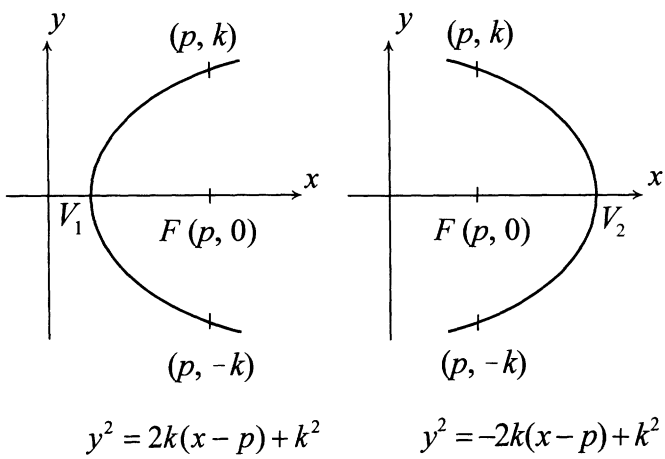

Figure 2.

To complete our discussion, we consider when the initial value problem (IVP) given by $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ has a unique solution. Suppose that $f(x, y)$ and $f_y(x, y)$ both are continuous on the region $R = \{(x, y) : a \le x \le b,\ c \le y \le d\}$ that contains (x_0, y_0) in its interior. Then there is an $h > 0$ and a unique function $y = y(x)$ defined on $(x_0 - h, x_0 + h)$ that is a solution to the IVP. (See, for example, [**1**, p. 13].) By the preceding remarks, it follows that the initial value problems

$$\frac{dy}{dx} = \frac{-(x-p) \pm \sqrt{(x-p)^2 + y^2}}{y} \quad \text{with} \quad y(x_0) = y_0\ (y_0 \neq 0)$$

have the functions defined by respective equations $y^2 = \pm 2k(x - p \pm \frac{k}{2})$ as their unique solutions. If $y_0 = 0$, the equations do not define functions on an open interval centered at x_0. However, the two equations $y^2 = \pm 2k(x - p \pm \frac{k}{2})$ with the condition $y(x_0) = 0$ yield the parabola $y^2 = 4(p - x_0)(x - x_0)$ that passes through $(x_0, 0)$.

We invite the reader to show that the ellipse and the hyperbola are characterized by their reflection properties by investigating the following exercises.

Exercises

1. Light rays emitted from the point $F_1\,(c, 0)$ $(c > 0)$ strike a plane curve C and reflect back through the point $F_2\,(-c, 0)$. Assuming that the angle of incidence equals the angle of reflection, show that the differential equation $xy(dy/dx)^2 + (x^2 - c^2 - y^2)(dy/dx) - xy = 0$ describes the shape of the curve C.
2. Show that for any $a > 0$, the curve $x^2/a^2 + y^2/(a^2 - c^2) = 1$ satisfies the equation in Exercise 1. Notice that for $a > c$, the curve is an ellipse with foci $F_1\,(c, 0)$ and $F_2\,(-c, 0)$. And for $a < c$ the curve is a hyperbola with the same foci.
3. Discuss the uniqueness of the solutions in Exercise 2, for the differential equation in Exercise 1.

The method we used to solve the differential equation (1) does not work for the equation in Exercise 1. Can you find a way to solve it?

References

1. Bernard Banks, *Differential Equations with Graphical and Numerical Methods*, Prentice Hall, 2001.
2. R. E. Johnson, and F. L. Kiokemeister, *Calculus and Analytic Geometry* (3rd ed.), Allyn and Bacon, 1964.

Sums of Roots and Poles of Rational Functions

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Consider rational functions $\frac{p(x)}{q(x)}$ expressed as $\frac{p(x)}{q(x)} = Q(x) + \frac{R(x)}{q(x)}$. For the examples

$$\frac{x^3 + 3x^2}{x^2 - x - 2} = x + 4 + \frac{6x + 8}{x^2 - x - 2} \quad \text{and} \quad \frac{6x^2 - 5}{2x + 1} = 3x - \frac{3}{2} - \frac{7/2}{2x + 1},$$

we see that the sum of Q's roots equals the sum of p's roots minus the sum of q's roots. This illustrates an interesting tidbit concerning rational functions that probably falls under the category of a known, but not well-known, fact among those of us in the trenches.

Let $p(x)$ and $q(x)$ be polynomials with no common factors and with real coefficients, where the respective degrees of p and q are n and m with $n > m \geq 0$. Using polynomial long division, we can write $\frac{p(x)}{q(x)} = Q(x) + \frac{R(x)}{q(x)}$ with the degree of $R(x)$ strictly less than the degree of $q(x)$. Then the sum of Q's roots equals the sum of p's roots minus the sum of q's roots.

Proof. Since we are interested only in the roots and poles, we may assume that the leading coefficients of $p(x)$ and $q(x)$ (and hence of $Q(x)$ also) are unity. Let

$$p(x) = \prod_{k=1}^{n} (x - a_k) = x^n - \left(\sum_{k=1}^{n} a_k\right) x^{n-1} + O(x^{n-2}) = x^n - Ax^{n-1} + O(x^{n-2}),$$