Return of the Grazing Goat in n Dimensions

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We return to the problem of the grazing goat considered by Marshall Fraser in CMJ 15 (March 1984) 126–134:

A goat is tethered to the edge of a disc shaped field of radius r. The goat's rope is of length kr. If the field is n-dimensional, what fraction of it can the goat reach, and what happens as n approaches infinity?

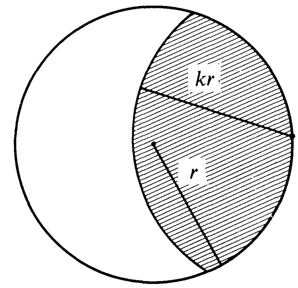

Figure 1

Our objective here is twofold: (a) To describe what happens to the proportion of the field which the goat can reach as n increases, where k is arbitrary but fixed; (b) to give a correct proof, in place of Fraser's incorrect one that if k_n is the tether length for the goat to reach half the field in dimension n, then $\lim_{n \to \infty} k_n = \sqrt{2}$. In fact, we establish

Theorem 1. *Suppose the goat's tether is a fixed proportion k of the n-disc radius. As n approaches infinity, the proportion of the volume which the goat can reach approaches zero if* $k < \sqrt{2}$, *one half if* $k = \sqrt{2}$, *and one if* $k > \sqrt{2}$.

The Grazing Goat with Fixed Tether Length. Since all the following will only refer to proportions, we can simplify by using unit discs for fields (i.e., $r = 1$). We shall thus write V_n for $V_n(1)$, the n-dimensional volume of an n-dimensional disc of radius 1. The region which the tethered goat can reach divides into two segments as drawn: the segment nearer to the tethering point, with volume $N_n(k)$, and the farther segment with volume $F_n(k)$. By similar triangles, we calculate the length $k^2/2$ as indicated.

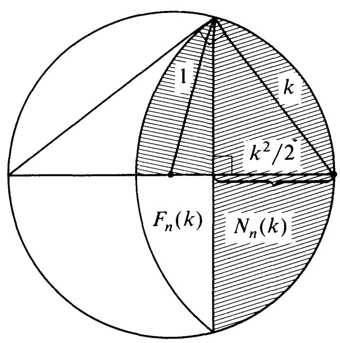

Figure 2

Now fix $k (0 \leqslant k \leqslant 2)$, and consider what happens to the proportion of the total volume which is in each segment. First we consider the nearer segment. Since

$$N_n(k) = \int_{1-k^2/2}^{1} V_{n-1}\left(\sqrt{1-x^2}\right) dx = V_{n-1} \int_{1-k^2/2}^{1} (1-x^2)^{(n-1)/2}\, dx,$$

we let $x = \sin\theta$ to get

$$N_n(k) = V_{n-1}\int_\alpha^{\pi/2}\cos^n\theta\, d\theta,$$

where $\sin\alpha = 1 - k^2/2$.

If $0 \leqslant k < \sqrt{2}$, then $0 < \alpha \leqslant \pi/2$ and we choose β with $0 < \beta < \alpha$. Hence,

$$N_n(k)/V_n = \int_\alpha^{\pi/2}\cos^n\theta\, d\theta\Big/\left(2\int_0^{\pi/2}\cos^2\theta\, d\theta\right) < (\pi/2)\cos^n\alpha/(2\beta\cos^n\beta)$$

$$= (\cos\alpha/\cos\beta)^n\pi/(4\beta)\to 0.$$

If $k = \sqrt{2}$, then $\alpha = 0$ and $N_n(k)/V_n = 1/2$ for all n. If $\sqrt{2} < k \leqslant 2$, then the part of the n-disc not covered by the nearer segment is the type of segment considered in Case 1, with a volume of $N_n'(k)$. So $N_n(k)/V_n = 1 - N_n'(k)/V_n \to 1$. For the farther segment, we shall now show that $F_n(k)/V_n \to 0$ for all k.

If $k \neq \sqrt{2}$, then $F_n(k)/V_n \int_{k^2/2}^{k} V_{n-1}(\sqrt{k^2 - x^2})\,dx/V_n$. Using $x = k\sin\theta$, this becomes

$$k^n\int_\alpha^{\pi/2}\cos^n\theta\, d\theta\Big/\left(2\int_0^{\pi/2}\cos^n\theta\, d\theta\right),$$

where $\sin\alpha = k/2$. Note that $0 \leqslant k^2 - k^4/4 < 1$ for this k (since the polynomial achieves its maximum of 1 at $\pm\sqrt{2}$). So there is a β with $0 < \beta < \pi/2$ and $\cos\beta > \sqrt{k^2(1 - k^2/4)}$. Now $\cos\alpha = \sqrt{1 - k^2/4}$; so

$$F_n(k)/V_n < (\pi/2)k^n\cos^n\alpha/(2\beta\cos^n\beta) = \left(\sqrt{k^2(1 - k^2/4)}/\cos\beta\right)^n\pi/(4\beta)\to 0.$$

For the case where $k = \sqrt{2}$, we consider a cone-shaped region containing the far segment (see Figure 3).

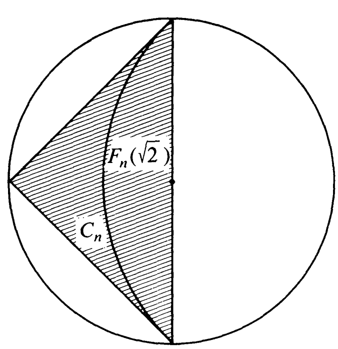

Figure 3

This region has volume

$$C_n = \int_0^1 V_{n-1}(x)\,dx = V_{n-1}\int_0^1 x^{n-1}\,dx = V_{n-1}/n.$$

Thus $V_n/C_n = 2n\int_0^{\pi/2}\cos^n\theta\,d\theta$. Integrating by parts, we get

$$\int_0^{\pi/2}\cos^n\theta\,d\theta = \cos^{n-1}\theta\sin\theta\big|_0^{\pi/2} + (n-1)\int_0^{\pi/2}\cos^{n-2}\theta\sin^2\theta\,d\theta$$

$$= (n-1)\int_0^{\pi/2}\cos^{n-2}\theta\,d\theta - (n-1)\int_0^{\pi/2}\cos^n\theta\,d\theta.$$

It follows that

$$V_n/C_n = 2n(n-1)/n\int_0^{\pi/2}\cos^{n-2}\theta\,d\theta = (n-1)/(n-2)(V_{n-2}/C_{n-2}).$$

Now let $a_n = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots(2n-1)/2n$ and $b_n = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots 2n/(2n+1)$. Then $C_n/V_n = b_{(n-2)/2}(C_2/V_2)$ for even n $(n > 2)$, and $C_n/V_n = a_{(n-1)/2}(C_1/V_1)$ for odd n. Since a_n and b_n are positive and decreasing, they converge to some a and b, respectively. Since $a_nb_n = 1/(2n+1)\to 0$, and $a_nb_n\to ab$, we must have a or b equal to 0. But $0 < a_n < b_n < 2a_n$, so $a = b = 0$. Thus, $C_n/V_n \to 0$. ■

The Grazing Goat Fallacy. For $0 \leqslant k \leqslant 2$, let $g_n(k)$ be the proportion of the unit n-disc which our goat can reach with a tether of length k. Thus, as n increases, $g_n(k) = (N_n(k) + F_n(k))/V_n$ converges to 0, 1/2, or 1 depending on whether $k < \sqrt{2}$, $k = \sqrt{2}$, or $k > \sqrt{2}$, respectively.

In Fraser's article, an incorrect proof is given of the true statement that the values k_n such that $g_n(k_n) = 1/2$ approach $\sqrt{2}$ as n increases. The fallacious part of the proof is that $g_n(\sqrt{2})\to 1/2$ and $g_n(k_n) = 1/2$ imply that $k_n \to \sqrt{2}$. To see that the above claim is insufficient, suppose $g_n(x) = x/n + 1/2$. Then $k_n = 0$ for all n since $g_n(0) = 1/2$. And $g_n(\sqrt{2})\to 1/2$. But certainly $k_n \not\to \sqrt{2}$. The difficulty is that more information is needed about g_n. This is just what has been provided, however, by Theorem 1. So we can now conclude with the following.

Theorem 2. *If the goat can reach a fixed proportion p $(0 < p < 1)$ of each unit n-disc, then the length of the tether converges to $\sqrt{2}$ as the dimension increases.*

Proof. For each n, since g_n is strictly increasing from 0 to 1, there is a unique k_n with $g_n(k_n) = p$. Now given any ϵ $(0 < \epsilon < 1/2)$, we have $g_n(\sqrt{2} - \epsilon)\to 0$ and $g_n(\sqrt{2} + \epsilon)\to 1$. Thus, there is an N such that $|k_n - \sqrt{2}| < \epsilon$ for $n \geqslant N$. Therefore, $k_n \to \sqrt{2}$. ■

Editor's Note: A number of readers were kind enough to write that $V_5(1)$ is the largest volume of the unit n-discs, not $V_6(1)$ as stated in the original article.