Since the Taylor series

$$2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3 + \sum_{n=4}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdot\dots\cdot(2n-3)}{2^{3n-1}n!}(x-4)^n$$

converges to $\sqrt{x}$ for $x \in (0, 8)$, the approximating polynomial will be an overestimate or an underestimate of $\sqrt{x}$ depending on whether the sign of the truncated part of the series is negative or positive, respectively. But, in a sufficiently small neighborhood of 4, the sign of the truncated part must be the sign of the first truncated term, $(1/512)(x-4)^3$. (Proving this fact or a more general version of it is a nice exercise in series and series convegence.) Since the exponent 3 is odd, this first truncated term will change sign from negative to positive at $x = 4$, so the polynomial is an overestimate of $\sqrt{x}$ to the left of $x = 4$ and an underestimate to the right, i.e., the graph of the approximating second degree polynomial does indeed cross the graph of $y = \sqrt{x}$.

One has, then, the following general rule for any function $y = f(x)$ that can be represented in a neighborhood of $x = a$ by its Taylor Series. If the function is approximated by a polynomial of degree m whose coefficients are chosen so that the polynomial and its first m derivatives agree with the function at $x = a$, and if n is such that $b_n = f^{(n)}(a)$ is the first non-zero derivative value for $n > m$ (there must be such, if a non-polynomial function is represented by its Taylor series), then the graph of the resulting approximating polynomial crosses the graph of the function at $x = a$ if and only if n is odd.

Barrow's Fundamental Theorem

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Somewhere around the middle of the first year of calculus we present our students with the Fundamental Theorem of Calculus, clothed in the majesty of capital letters and demonstrated with the genius of three hundred years of great mathematicians. It is, however, a remarkable fact that its fundamental geometric content is accessible directly from high school plane geometry.

When Sir Isaac Newton said, toward the end of his life, "If I have seen further than others, it is because I have stood on the shoulders of giants," his mentor, Isaac Barrow, who preceded him in the Lucasian chair of mathematics must certainly have been on his mind. In 1670, Newton edited and published Barrow's *Lectiones Geometricae*. They contained, in Lecture X, the following theorem, here presented in modern terminology, which expresses the geometric content of the Fundamental Theorem of Calculus.

Referring to Figure 1, let f be a strictly increasing, positive, continuous function on $[a, b]$ and let F be the function whose value at each x in $[a, b]$ is the area under the graph of f from a to x. For convenience, we draw the graph of $-f$ instead of the graph of f. From any point, A, on the x-axis between $x = a$ and $x = b$, erect a perpendicular. Let K be its intersection with the graph of F and B be its intersection with the graph of $-f$.

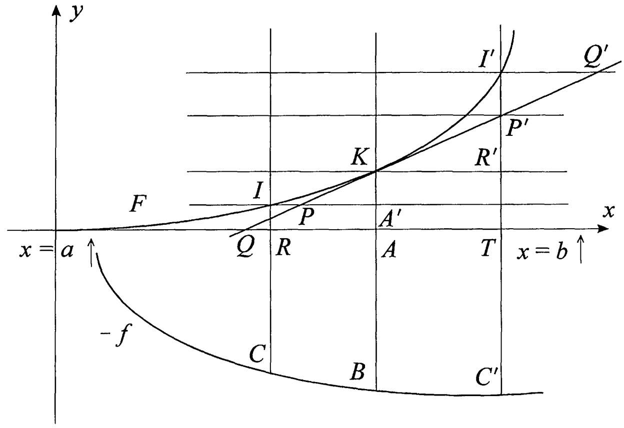

Figure 1.

Theorem. If a line is drawn from the point K to a point on the x-axis, Q, so that

$$\overline{AK}/\overline{AQ} = \overline{AB}$$

(this can always be done with ruler and compass), then this line will be tangent at K to the graph of F.

Proof. Barrow understood a tangent to be a line that intersects a curve at only one point. The proof, therefore, consists in showing that the line $\overline{QK}$ is below, or to the right of, the graph of F.

Again referring to Figure 1, $\overline{AB} = \frac{\overline{AK}}{\overline{AQ}} = \frac{\overline{A'K}}{\overline{A'P}}$ by similar triangles. $\overline{AB} \cdot \overline{A'P} = \overline{A'K} =$ area $CRAB$ since, by hypothesis, the area under the graph of f on an interval $[r, s]$ is $F(s) - F(r)$. And, area $CRAB < \overline{AR} \cdot \overline{AB}$ since f is strictly increasing. Therefore, $\overline{A'P} < \overline{AR}$, which means that the point P on the line $\overline{KQ}$ is to the right of the point I on the graph of F.

Similarly, $\overline{AB} = \frac{\overline{P'R'}}{\overline{KR'}}$ by similar triangles. $\overline{P'R'} = \overline{AB} \cdot \overline{KR'} <$ area $C'TAB = \overline{I'R'}$.

Therefore the point P' on the line $\overline{KQ'}$ (the extension of the line $\overline{KQ}$) is strictly below the point I' on the graph of F.

Students may be tempted to conclude that we have shown that

$$\frac{d}{dx}\int_0^x f(u)\,du = f(x)$$

from which, of course, the Fundamental Theorem is a short hop. However, what has actually been shown is purely geometric. There is no analytic content. There is no computational algorithm for determining the area under f, or the slope of the tangent line to the graph of F. It was Newton's genius to create, on this geometric skeleton, the muscle of computational methods which, axiomatized and extended, we call calculus.

References

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