Proving Heron's Formula Tangentially

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To Heron of Alexandria (circa A.D. 200), is attributed the formula

$$K = \sqrt{s(s-a)(s-b)(s-c)} \qquad \left(s = \frac{a+b+c}{2}\right) \tag{1}$$

for the area of a triangle ABC with sides a, b, c. For an elementary proof of (1), consider the inscribed circle whose center is at I and whose radius is r. Clearly,

$$K = rs \tag{2}$$

follows by summing the areas $\frac{ra}{2} + \frac{rb}{2} + \frac{rc}{2}$ of triangles IBC, ICA and IAB, respectively. Our objective will be to establish (1) by an interesting expression for r.

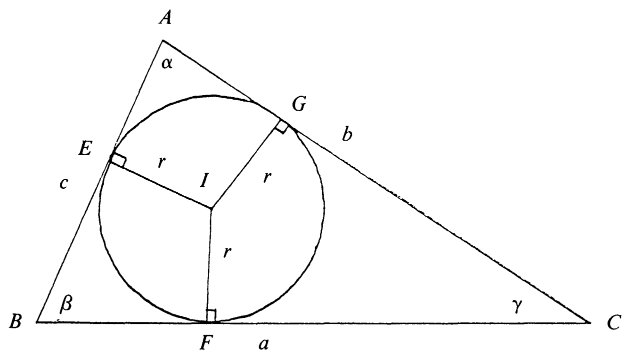

Figure 1.

Since tangents to a circle from a point are equal, we have

$$EB = s-b, \quad CG = s-c, \quad \text{and} \quad AE = s-a. \tag{3}$$

And since I is the point of intersection of the bisectors of the (internal) angles of triangle ABC, we can use (2) to obtain

$$\tan\left(\frac{\beta}{2}\right) = \frac{r}{s-b}, \quad \tan\left(\frac{\gamma}{2}\right) = \frac{r}{s-c}, \quad \text{and} \quad \tan\left(\frac{\alpha}{2}\right) = \frac{r}{s-a}. \tag{4}$$

From $\alpha + \beta = \pi$ and the fact that $\tan(\frac{\pi}{2} - x) = \frac{1}{\tan x}$, it follows that

$$\tan\frac{\gamma}{2} = \tan\left\{\frac{\pi}{2} - \left(\frac{\alpha}{2} + \frac{\beta}{2}\right)\right\} = \frac{1 - \tan\left(\frac{\alpha}{2}\right)\cdot\tan\left(\frac{\beta}{2}\right)}{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)}. \tag{5}$$

Substituting (4) into (5), we obtain

$$\frac{r}{s-c} = \frac{(s-a)(s-b) - r^2}{r\left[(s-b) + (s-a)\right]}. \tag{6}$$

This can be rewritten as

$$r^2(s - b + s - a) = (s - a)(s - b)(s - c) - r^2(s - c). \tag{7}$$

In like manner,

$$\frac{\alpha}{2} = \frac{\pi}{2} - \left(\frac{\beta}{2} + \frac{\gamma}{2}\right) \quad \text{and} \quad \frac{\beta}{2} = \frac{\pi}{2} - \left(\frac{\alpha}{2} + \frac{\gamma}{2}\right)$$

yield

$$r^2(s - c + s - b) = (s - a)(s - b)(s - c) - r^2(s - a) \tag{8}$$

and

$$r^2(s - a + s - c) = (s - a)(s - b)(s - c) - r^2(s - b), \tag{9}$$

respectively. Adding (7), (8), (9), and recalling that $2s = a + b + c$, we find

$$r^2 = \frac{(s - a)(s - b)(s - c)}{s}. \tag{10}$$

Thus, substitution of (10) into (2) yields (1).

Editor's Note: There are numerous proofs of Heron's formula which do not refer to an inscribing circle. In Ivan Niven's *Maxima and Minima Without Calculus* (MAA Dolciani Series, No. 6 (1981), 7–11], for example, the derived formula for the area of a quadrilateral with sides a, b, c, d is

$$A^2 = (s - a)(s - b)(s - c)(s - d) - \frac{abcd}{2}[1 + \cos(\theta + \lambda)],$$

where $s = (a + b + c + d)/2$ and θ, λ are a pair of interior angles not sharing a common side. When the quadrilateral is inscribable in a circle, $\theta + \lambda = \pi$ and so we obtain Brahmagupta's formula

$$A = \sqrt{(s - a)(s - b)(s - c)(s - d)}.$$

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Income Tax Averaging and Convexity

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To compute the income tax for a given year, one method that can be used is income averaging. Roughly, this method is applicable when your taxable income for the current year is "much larger" than that for each of the four previous years. For instance, income averaging is often appropriate for college graduates who have completed the first year of their new job after having supported themselves with grant/scholarship money and part-time jobs during college.

To quote from the Internal Revenue Service Publication 17 [Your Federal Income Tax (For Individuals) for Use in Preparing 1982 Returns]: "If your income has increased substantially this year, it may be to your advantage to figure your tax under the income averaging method. Under this method, part of an unusually large amount of taxable income can be taxed at lower rates, and this may result in a