

note, it is shown that the formulas arise naturally in connection with the quadratic formula.

The main idea is to apply the quadratic formula to  $(x - r)(x - s) = 0$ , attempting to recover the roots,  $r$  and  $s$ . Therefore, writing the equation in the standard form

$$x^2 - (r + s)x + rs = 0,$$

the quadratic formula gives the roots as

$$\frac{1}{2}(r + s \pm \sqrt{(r + s)^2 - 4rs}) = \frac{1}{2}(r + s \pm |r - s|).$$

Clearly,  $\frac{1}{2}(r + s + |r - s|)$  is the larger root (i.e., the larger of  $r$  and  $s$ ) and  $\frac{1}{2}(r + s - |r - s|)$  is the smaller root (i.e., the smaller of  $r$  and  $s$ ).

There are several features of this derivation that are beneficial in the classroom. First, having derived formulas for obtaining roots from coefficients, and vice versa, it is appropriate to observe the inverse nature of the procedures, and to verify the conclusion by composition. Second, the derivation provides an opportunity to remind students of the identity  $\sqrt{x^2} = |x|$  in a setting where the absolute value function has obvious significance. (For example, if  $\sqrt{(r - s)^2}$  is simply replaced by  $(r - s)$ , we appear to find that  $r$  is always the larger root!) Finally, the formulas (1) and (2) are of intrinsic interest, and their derivation illustrates how the unexpected can pop up so fortuitously in mathematics.

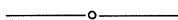
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*Editor's Note:* Formulas (1) and (2) can be extended to find the minimum and maximum of any  $n$  given numbers. For instance, students can now be asked to establish that

$$\max(r, s, t) = \frac{1}{2} \left[ \frac{1}{2}(r + s + |r - s|) + t + \left| \frac{1}{2}(r + s + |r - s|) - t \right| \right]$$

$$\min(r, s, t) = \frac{1}{2} \left[ \frac{1}{2}(r + s - |r - s|) + t - \left| \frac{1}{2}(r + s - |r - s|) - t \right| \right]$$

for given  $r, s, t$ .



### Probabilistic Repeatability Among Some Irrationals

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It is well known that most square roots are irrational: their decimal expansions are nonrepeating and nonterminating. Thus, intuition would have us suspect that each of the ten digits would be equally likely to appear in the  $r$ th place  $d_r$  beyond the decimal point in a given  $\sqrt{x}$  ( $x > 0$ ). That such is not the case is the point of this note.

*If  $b$  is a nonnegative integer, and  $x$  is chosen at random in the closed interval from  $b^n$  to  $(b + 1)^n$ , what is the probability  $P(d_r, k)$  that the  $r$ th digit  $d_r$  in the decimal representation of  $x^{1/n}$  is  $k$ ?*

For  $b$  and  $n$  given, let

$$B(d_r, k) = \{x \in [b^n, (b + 1)^n] : d_r \text{ in } x^{1/n} \text{ is } k\}.$$

By way of illustration, suppose  $b = 1$  and  $n = 2$ . Then

$$B(d_3, k) = \{x \in [1^2, 2^2] : \sqrt{x} = 1.d_1d_24d_5d_6 \cdots\}.$$

Since  $10^2x^{1/2} = 10^2 + d_1d_2 + .4d_4d_5$ , we see that  $x \in B(d_3, 4)$  if and only if there is an integer  $p \in [0, 10^2)$  such that  $10^2 + p + \frac{4}{10} \leq 10x^{1/2} < 10^2 + p + \frac{5}{10}$ . Clearly  $2 \in B(d_3, 4)$ , since the condition holds for  $p = 41$ .

In general,  $x \in B(d_r, k)$  if and only if there is an integer  $p \in [0, 10^{r-1})$  such that

$$10^{r-1}b + p + \frac{k}{10} \leq 10^{r-1}x^{1/n} < 10^{r-1}b + p + \frac{k+1}{10}. \quad (1)$$

This can be recast as

$$\frac{\left[10^{r-1}b + p + \frac{k}{10}\right]^n}{10^{n(r-1)}} \leq x < \frac{\left[10^{r-1}b + p + \frac{k+1}{10}\right]^n}{10^{n(r-1)}}. \quad (2)$$

Geometrically, (1) says that  $x \in B(d_r, k)$  if and only if  $10^{r-1}x^{1/n}$  belongs to one of the  $10^{n-1}$  intervals  $\left[10^{r-1}b + p + \frac{k}{10}, 10^{r-1}b + p + \frac{k+1}{10}\right]$  determined by the integral values of  $p \in [0, 10^{r-1} - 1]$ . This is equivalent to saying that  $B(d_r, k)$  is the union of the  $10^{r-1}$  intervals depicted in (2). Thus, assuming that  $x$  is uniformly distributed on the interval  $[b^n, (b+1)^n]$ , we have

$$P(d_r, k) = \sum_{p=0}^{10^{r-1}-1} \frac{\left\{\left(10^{r-1}b + p + \frac{k+1}{10}\right)^n - \left(10^{r-1}b + p + \frac{k}{10}\right)^n\right\}}{10^{n(r-1)}[(b+1)^n - b^n]}. \quad (3)$$

For  $n = 2$ , formula (3) reduces to

$$P(d_r, k) = \frac{.2b + \frac{.02k}{10^{r-1}} + \frac{10^{r-1} - .9}{10^r}}{2b + 1}. \quad (4)$$

Thus, the probabilities that  $\sqrt{x}$  has  $k$  as its  $r$ th digit  $d_r$  ( $r = 1, 2, 3$ ) are

$$P(d_1, k) = \frac{.2b + .02k + .01}{2b + 1},$$

$$P(d_2, k) = \frac{.2b + .002k + .091}{2b + 1},$$

and

$$P(d_3, k) = \frac{.2b + .0002k + .0991}{2b + 1}.$$

The following tables give  $P(d_r, k)$  for  $b = 0$  and  $n = 2, 3$ .

$n = 2$				$n = 3$			
$P(d_r, k)$	$r = 1$	$r = 2$	$r = 3$	$P(d_r, k)$	$r = 1$	$r = 2$	$r = 3$
$k = 0$	.01	.091	.0991	$k = 0$	.001	.08686	.0986536
$k = 1$	.03	.093	.0993	$k = 1$	.007	.08962	.0989512
$k = 2$	.05	.095	.0995	$k = 2$	.019	.09244	.0992494
$k = 3$	.07	.097	.0997	$k = 3$	.037	.09532	.0995482
$k = 4$	.09	.099	.0999	$k = 4$	.061	.09826	.0998476
$k = 5$	.11	.101	.1001	$k = 5$	.091	.10126	.10011476
$k = 6$	.13	.103	.1003	$k = 6$	.127	.10432	.1004482
$k = 7$	.15	.105	.1005	$k = 7$	.169	.10744	.1007494
$k = 8$	.17	.107	.1007	$k = 8$	.217	.11062	.1010512
$k = 9$	.19	.109	.1009	$k = 9$	.271	.11386	.1013536

As is clear from the above,  $P(d_r, k)$  is increasing in  $k$  for each fixed  $r$ ; larger digits are more likely to occur in any given decimal position  $d_r$ . This monotonicity is most pronounced for  $P(d_1, k)$ . For further references, see The law of Anomalous Numbers by Frank Benford, Proc. Amer. Phil. Soc., 78 (1938) 551–572, and “The First Digit Problem” by R. A. Raimi, The Amer. Math. Monthly, 83 (1976) 521–538.

The manner of de Moivre's death has a certain interest for psychologists. Shortly before it, he declared that it was necessary for him to sleep some ten minutes or a quarter of an hour longer each day than the preceding one: the day after he had thus reached a total of something over twenty-three hours he slept up to the limit of twenty-four hours, and then died in his sleep.

Walter William Rouse Ball (1850–1925), History of Mathematics (London: Macmillan) 1911.

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