The maximizing z is clearly independent of the shape and always $c/3$. However, the maximum volume $(4A/P^2)(c/3)^3$ does depend on the shape through the ratio $4A/P^2$. Indeed, the appearance of the isoperimetric ratio $4A/P^2$ in the solution clarifies the situation and opens up a whole new avenue for classroom discussion.

Note that two-thirds of c is left for the perimeter of the cross section so that maximum dimension of the cross section cannot exceed $c/3$, one-half of the perimeter. The largest dimension of the box is indeed what we have called the length.

One last question. What is the "correct" generalization to other dimensions?

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What is the Biggest Rectangle You Can Put Inside a Given Triangle?

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The purpose of this note is to put together an instructive package of pleasant established theorems dealing with certain maximum rectangles inside triangles and several other results of this flavor.

The following familiar old calculus book problem is dealt with in [**1**], [**3**], [**5**], [**6**], and [**7**] in one form or another: *Given a triangle of altitude a and base b, find the dimensions of the rectangle of maximum area that can be inscribed in this triangle with one side along the base.* (In many old books the triangle is a *right* triangle.)

We first remind ourselves of a solution of this problem. See Figure 1, where the two essentially different cases are shown; in (i) but not in (ii), the vertex C is "above" the base.

Starting with (i) in Figure 1, we seek the maximum of the product xy, where, from similar triangles, we have $y = (b/a)(a-x)$. Thus, $xy = (b/a)(x)(a-x)$, and we need to find x so that this is maximized. We need not use a derivative; we can complete the square to observe that $x(a-x) = (a/2)^2 - \{x-(a/2)\}^2$, and this is a

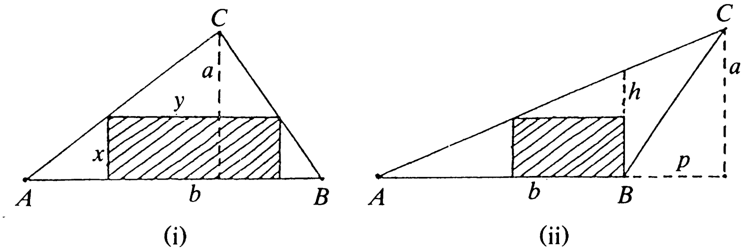

Figure 1

maximum if, and only if, $(x - a/2)^2 = 0$; i.e., iff $x = a/2$. It follows that

$$xy = (b/a)(x)(a-x) \le (1/2)(ab/2) = (1/2)\text{Area}(\triangle ABC),$$

with equality iff $x = a/2$. Thus, the maximum rectangle has height $x = a/2$ and area exactly half that of the given triangle.

(In [**2**], Cantor says that this result in Euclid Book VI Proposition 27 embodies (not only) the first maximum in the history of mathematics for which the proof is given, but is the first to be written as a function: $x(a-x)$ achieves its maximum value for $x = a/2$.)

In case (ii), the maximum rectangle will have height $h/2$ and area less than half that of the given triangle. (In this case, the maximum area is $\{b/(b+p)\}$ $(1/2)\text{Area}(\triangle ABC)$.)

We note that, in case (ii), if we chose AC as the base on which our rectangle rests, we would once more obtain a maximum rectangle with area half that of the given triangle. Thus, in any case, it is possible to inscribe *some* rectangle of area half that of the given triangle.

Again in one form or another in those same references (above)—and, who knows, perhaps even in the lost seminar notes of the sons and daughters of Adam around the time of Eden so long ago—this question is then posed: "*Are there cases where a larger rectangle can be placed in a triangle when we do not require that a side of the rectangle lie along a side of the given triangle*?"

The answer established in [**3**] can be stated this way:

Theorem. *Among all rectangles with vertices lying inside or on a given triangle T, there is at least one rectangle R of maximum area. If T is an obtuse-angled triangle, there is only one such rectangle; if T is a right triangle, there are two, and if T is acute-angled, there are three. The area of any rectangle R of maximum area is one-half the area of T, and each such rectangle R is in a special position, with one side lying on a side of T and the opposite side of R joining the midpoints of the other two sides of T.* (See Figure 2.)

My late colleague, M. T. Bird, found a beautifully simple and direct proof of a key part of that result. His sharp observation in [**1**] deserves to be better known. Briefly, here is what he did: He showed that, given any rectangle R in a triangle T, then there is a rectangle R' in that triangle such that R' has area at least as great as R, and such that R' has one of its sides along a side of T. Of course, this then shows that our discussion with Figure 1, above, applies and settles the question we asked. To show the principal elements of his simple argument, it suffices to consider Figure 3, where a rectangle R with sides of lengths a and b is inside a given triangle. We construct the rectangle R' inside the triangle as shown, drawing first the side whose length is called b', a side which is parallel to the base (below) of the given triangle. We then have $a/b' = a'/b$; that is, $ab = a'b'$, telling us that R has the same area as the rectangle R' *which rests on a side of the given triangle*. And furthermore, this latter rectangle, R', may even have to be stretched horizontally to meet the sides of the given triangle, yielding an even larger area. Of course, the observation about the role of the midpoints is a sub-observation of the work done in connection with Figure 1, above.

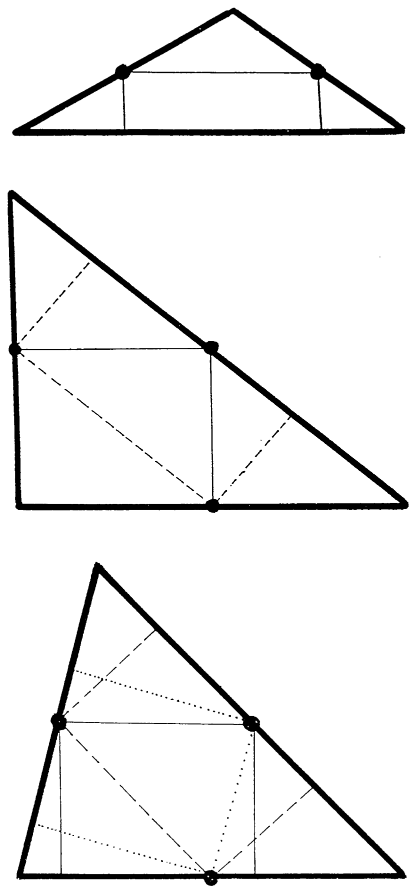

Figure 2

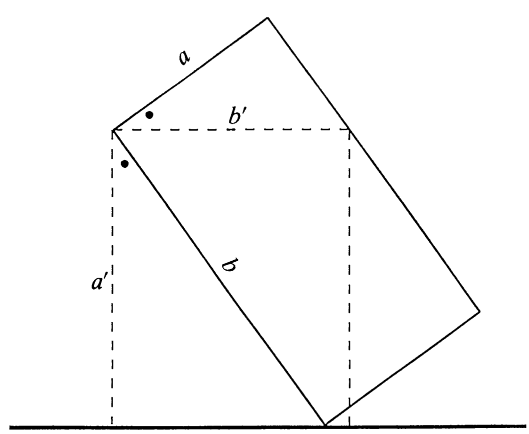

Figure 3

In **[5]**, the author presents the results of a series of problems for her students dealing with various orientations of rectangles inside a triangle—for example, those making a fixed angle with a side of the encompassing triangle. Her interesting results and procedures (employing the law of sines, in particular) repeatedly assign important roles to midpoints.

In this matter of midpoints it fits to add to our package another theorem from **[3]**: *Let K be a convex region and let $n \geq 3$ be a given integer. If P is a convex n-gon of minimal area containing K, then the midpoints of the sides of P lie on the boundary of K.* This last result has been shown by M. M. Day [4] to hold in three dimensions as well, with "midpoints of the sides of P" replaced by "centroids of the planes of P."

Some of these matters are treated in Chapter 3 of **[8]**, where the author cleverly establishes a result which belongs in our present package because it too deals with an "orientation" question: The sides of a rectangle inscribed in an ellipse are parallel to the axes of the ellipse.

In closing, here are two (not exactly trivial) homework problems: (1) Find the largest square that can be inscribed in some triangle of area 1; (2) Find the largest cube that can be inscribed in some tetrahedron of volume 1. [These are problems E2930 and E3114, respectively, from the *American Mathematical Monthly*. For their respective solutions, see Vol 91 (1984) pp. 141–142; and Vol 95 (1988) p. 55.]

Finally, pedagogical duty requires the mention of two bits of good advice from the master teacher George Pólya **[9]**, for the benefits of following his simple-sounding but undeniably powerful advice are illustrated above in several ways:

(a) When you have found a result, see if you can establish it in another way;
(b) Look around when you have got your first mushroom or made your first discovery; they grow in clusters.

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