that $\frac{g(f(x)) - g(f(c))}{x - c}$ can be made arbitrarily small in a sufficiently small neighborhood of c, so the defining limit exists and is zero, proving the theorem.

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The Average Distance of the Earth from the Sun

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In astronomy texts one sometimes finds a description of an Astronomical Unit (AU) as the "average distance of the earth from the sun." However, consideration of the concept of the average, or mean value, of a function reveals that the parameterization (or, equivalently, the underlying metric) must be specified in order to have a precise definition of that mean value. A more careful definition of AU is as the length of the semi-major axis of the earth's orbit. In this note we will show how the choice of parameter from among several plausible alternatives can affect the average value of the distance.

In order to investigate this topic we recall that the average value of the function f over the interval $[a, b]$ is given by

$$Avg(f, t) = \frac{1}{b - a}\int_a^b f(t)\, dt \tag{1}$$

and that, if $\tau = \varphi(t)$ is a change of parameter with $\alpha = \varphi(a)$ and $\beta = \varphi(b)$, then

$$Avg(f, \tau) = \frac{1}{\beta - \alpha}\int_{\tau=\alpha}^{\tau=\beta} f(\varphi^{-1}(\tau))\, d\tau = \frac{\int_{t=a}^{t=b} f(t)\,\varphi'(t)\, dt}{\int_{t=a}^{t=b} \varphi'(t)\, dt} = \frac{\int_{t=a}^{t=b} f(t)\, d\tau}{\int_{t=a}^{t=b} d\tau}$$

gives its value with respect to the new parameter.

In the following examples we will consider the polar equation of an ellipse,

$$r = \frac{1 - e^2}{1 + e\cos\theta} \tag{2}$$

with semi-major axis of length 1 and eccentricity e, and with one focus as the origin of the coordinate system. This equation could (if $e = 0.0167$) represent the distance in Astronomical Units between the earth and the sun (at the origin). We will examine various ways of calculating the average value of r.

1. **Average with respect to angle of revolution about the sun.** This average follows from (2) and the average value formula (1); it is the usual average obtained from (2).

$$Avg(r, \theta) = \frac{1-e^2}{2\pi} \int_0^{2\pi} \frac{d\theta}{1 + e\cos\theta}.$$

2. **Average with respect to distance traveled.** This is the arc length parameterization.

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = (1-e^2)\frac{\sqrt{1+2e\cos\theta + e^2}}{(1+e\cos\theta)^2}\, d\theta,$$

$$Avg(r, s) = \frac{\int_{\theta=0}^{\theta=2\pi} r\,ds}{\int_{\theta=0}^{\theta=2\pi} ds} = (1-e^2)\frac{\int_0^{2\pi} \frac{\sqrt{1+2e\cos\theta + e^2}}{(1+e\cos\theta)^3}\, d\theta}{\int_0^{2\pi} \frac{\sqrt{1+2e\cos\theta + e^2}}{(1+e\cos\theta)^2}\, d\theta}.$$

3. **Average with respect to time.** Since Kepler's Second Law tells us that orbital time is proportional to area swept out, we can calculate this average using area as our parameter. Thus

$$dA = \frac{1}{2} r^2\, d\theta$$

giving

$$Avg(r, A) = \frac{\int_{\theta=0}^{\theta=2\pi} r\,dA}{\int_{\theta=0}^{\theta=2\pi} dA} = (1-e^2)\frac{\int_0^{2\pi} \frac{d\theta}{(1+e\cos\theta)^3}}{\int_0^{2\pi} \frac{d\theta}{(1+e\cos\theta)^2}}.$$

4. **Average with respect to projection onto the major axis.** We calculate this by representing the ellipse centered in rectangular coordinates. Using the top half of the ellipse

$$y^2 = (1-e^2)^2(1-x^2),$$

$$r = \sqrt{(x-e)^2 + y^2},$$

$$Avg(r, x) = \frac{1}{2}\int_{-1}^{1} r\,dx = \frac{1}{2}\int_{-1}^{1} \sqrt{(x-e)^2 + (1-e^2)^2(1-x^2)}\, dx.$$

Example values. We consider ellipses with eccentricities equal to those of the orbits of the earth, Pluto, and Halley's Comet. For ease of comparison we use a semi-major axis length of 1 for each case. The following values were calculated using built-in numeric integration on a TI-92 calculator.

Table 1. Average distance from the sun of various planets, using semi-major axis length as the unit length and various parameterizations.

		Earth	Pluto	Halley's Comet
Eccentricity		0.0167	0.2481	0.97
Average by	Angle	0.999861	0.968734	0.243105
	Arc Length	1.000000	1.000000	1.000000
	Time	1.000139	1.030777	1.470450
	Projection	0.999907	0.980260	0.972443

Observations and Conclusions

Not surprisingly, the greater the eccentricity the greater the differences arising from the different methods. It may not have been expected that the arc length representation would always produce the length of the semi-major axis, as it did, but this will be demonstrated below. As a consequence it is correct to describe an Astronomical Unit as the average *relative to arc length* of the earth's distance from the sun.

Invariance over eccentricity of the arc length based average

In Figure 1 we offer a proof without words that the average distance using the arc length parameterization is always equal to the length of the semi-major axis.

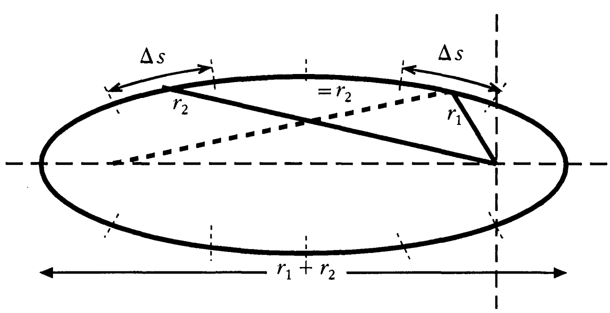

Figure 1. Average under arc length parameter.