To prove this, if we use the same technique as before with $I + A$ in place of A, we see that the equation of $(I + T)U$ is

$$\left((1 + d^2) + c^2\right) x^2 - 2(b + bd + c + ca)xy + \left((1 + a)^2 + b^2\right) y^2 = (1 + a + d + ad - bc)^2 \qquad (2)$$

which can be reduced to diagonal form by rotation of axes. The rotations needed to diagonalize TU and $(I + T)U$ will be the same exactly when the two symmetric matrices B and C of the quadratic forms (1) and (2) are simultaneously diagonalizable. Since they are symmetric, they are simultaneously diagonalizable if and only if they commute [1]. Since B and C are scalar multiples of $(A^{-1})^t(A^{-1})$ and $((I + A)^{-1})^t(I + A)^{-1}$ respectively, they commute when AA^t and $A + A^t$ commute, which happens exactly when $AA^t(A + A^t)$ is symmetric. Comparing the off-diagonal entries of $AA^t(A + A^t)$ we see that it is symmetric if and only if $(c - b)((a - d)^2 + (b + c)^2) = 0$. Thus B and C are simultaneously diagonalizable if and only if either $b = c$ or $a = d$ and $c = -b$. When this occurs, the same matrix diagonalizes both B and C.

If A is of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ then A is a scalar multiple of an orthogonal matrix. In this case it is easy to see that both TU and $(I + T)U$ are circles with radii $\det A$ and $\det(I + A)$ respectively.

The examples used before illustrate the geometry. $A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$ is symmetric so the axes of TU and $(I + T)U$ are the same (Fig. 2a) but $A = \begin{pmatrix} .5 & 0 \\ .5 & 1 \end{pmatrix}$ is not one of the proper forms and the axes of TU and $(I + T)U$ are different (Fig. 2b).

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Convergence-Divergence of p-Series

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There are many arguments to show that the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges, including two recent proofs by contradiction by Ecker [**3**] and by Cusumano [**2**] (see also Cohen and Knight [**1**]). But the more general problem of determining the convergence or divergence of the p-series $\sum_{n=1}^{\infty} 1/n^p$ is almost always solved by using some form of the integral test. Here, using methods inspired by [**2**], we show how the problem can be solved without using the integral test.

We consider first the divergence problem. Suppose that $p \leq 1$ and $\sum_{n=0}^{\infty} 1/(n+1)^p$ is convergent, say to L. Then

$$L = \sum_{n=0}^{\infty} \frac{1}{(n+1)^p} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^p} + \sum_{n=0}^{\infty} \frac{1}{(2n+2)^p}$$

$$= \sum_{n=0}^{\infty} \frac{2}{(2n+2)^p} + \sum_{n=0}^{\infty} \left(\frac{1}{(2n+1)^p} - \frac{1}{(2n+2)^p} \right) = 2^{1-p} L + Q,$$

where $Q = \sum_{n=0}^{\infty} (1/(2n+1)^p - 1/(2n+2)^p) > 0$. Thus $L(1 - 2^{1-p}) = Q$.

If $p = 1$, then $Q = 0$, which is impossible (or see the observation of Cusumano [**2**]). If $p < 1$, then $Q < 0$, which also is impossible. Hence the assumption that $L < \infty$ for $p \leq 1$ leads to a contradiction, and so the series diverges for $p \leq 1$.

Now we consider the convergence problem. Suppose that $p > 1$ and set $s_m = \sum_{n=0}^{m} 1/(n+1)^p$. Letting $N = 2m$ we see that

$$s_N = \sum_{n=0}^{m} \frac{1}{(2n+1)^p} + \sum_{n=0}^{m-1} \frac{1}{(2n+2)^p}$$

$$= \sum_{n=0}^{m} \frac{2}{(2n+2)^p} - \frac{1}{2^{p-1}(m+1)^p} + \sum_{n=0}^{m} \left(\frac{1}{(2n+1)^p} - \frac{1}{(2n+2)^p} \right)$$

$$= \frac{1}{2^{p-1}} s_m - \frac{1}{2^{p-1}(m+1)^p} + Q,$$

where $Q = \sum_{n=0}^{m} (1/(2n+1)^p - 1/(2n+2)^p)$. Since

$$s_N = s_m + \sum_{n=m+1}^{2m} 1/(n+1)^p,$$

this gives

$$s_m = \frac{1}{2^{p-1}} s_m - \frac{1}{2^{p-1}(m+1)^p} - \sum_{n=m+1}^{2m} \frac{1}{(n+1)^p} + Q$$

which implies that

$$\left(1 - \frac{1}{2^{p-1}}\right) s_m \leq Q.$$

But

$$Q = 1 - \left(\frac{1}{2^p} - \frac{1}{3^p}\right) - \cdots - \left(\frac{1}{(2m)^p} - \frac{1}{(2m+1)^p}\right) - \frac{1}{(2m+2)^p} \leq 1,$$

so $s_m \leq \frac{1}{1 - 1/2^{p-1}}$. Thus the sequence of partial sums is monotonic and bounded and hence converges, and the p-series converges for $p > 1$.

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Linear Algebra in the Financial World

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In the beginning of 1999 some financial funds bought call options on gold with an exercise price of \$390 an ounce and exercise date December 1999. This means that they had the option to buy gold at \$390 an ounce in December. The reason for entering into the contract was the expectation (based on the millennium fever and Y2K scare) that the price of gold would rise dramatically in December 1999. For example, if the price did rise to \$400 an ounce, the pay-off would be \$10 an ounce. The price of each call option was \$1 per ounce, so the profit on 10,000 call options, would have been 10,000 × (\$400 − \$390) − 10,000(\$1) or \$90,000! If the price remained below \$390, the option would not be exercised and \$10,000 lost. How was the price of \$1 per call option determined so that the contract was fair to both parties? Can all possible contracts be priced fairly? These important problems involve concepts like arbitrage and complete markets, and this is what I want to discuss with the aid of linear algebra pictures as in [**3**].

Some Financial Concepts

Consider a discrete time single period model of a market where we have N assets. Their prices are given by numbers $S_1^0, S_2^0, \ldots, S_N^0$ at the initial time $t = 0$ and their prices at some future time $t = 1$ are random variables $S_1^1, S_2^1, \ldots, S_N^1$. If we allow *arbitrage* in a model of a market in equilibrium, then it means you could start with no money and have a chance of ending up with some money with no risk at all. Since this is not a reasonable situation, we assume there are no arbitrage opportunities. It can be proved [**2**] that this is equivalent to the existence of a risk neutral probability measure Q. What do we mean by a risk neutral probability measure? Well, if there are K possible states of the world $\omega_1, \ldots, \omega_K$, that can be realized at a certain time in the future, then $Q = (Q(\omega_1), \ldots, Q(\omega_K))$ where $Q(\omega_i) > 0$ for $i = 1, \ldots, K$ and $\sum_{i=1}^{K} Q(\omega_i) = 1$. Furthermore, the expectation with respect to Q of the future asset prices S_j^1, is $(1+r)S_j^0$ where r is the interest rate. Roughly speaking, Q is that measure which ensures that *on average* you can do no better with asset j than you would have done by initially depositing amount S_j^0 in the bank. (In a specific state of the world you may of course do better). This Q ensures that claims or contracts on these N assets are priced fairly for both parties entering into the contract. Mathematically:

$$E_Q[Sj] = \sum_{k=1}^{K} Q(\omega_k) S_j^1(\omega_k) = (1+r)S_j^0, \qquad \text{for each } j = 1, \ldots, N$$

