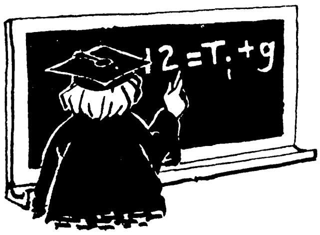

CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

Readers are invited to submit material for consideration to:

Warren Page
Department of Mathematics
New York City Technical College
300 Jay Street
Brooklyn, N.Y. 11201

Pythagorean Systems of Numbers

Joseph Wiener, Pan American University, Edinburg, TX

The task of identifying all solutions in the positive integers of the equation $x^2 + y^2 = z^2$ was raised in the days of the Babylonians and was a favorite with the ancient Greek geometers. It was not until Euclid wrote his *Elements* that a complete solution to the problem appeared. Today, although the characterization of all primitive Pythagorean triples is fairly well known, there still continues to be interest in the study of new ways of generating Pythagorean triples and quadruples.

In this note, we offer another approach to the problem of constructing Pythagorean systems with any number of terms. This elementary approach is rather attractive since it encompasses such special cases as

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = y_n^2,$$

in which the sum of the first two squares is a square, the sum of the first three squares is a square, etc. We shall also use our approach to solve an interesting problem concerning the squares of certain consecutive natural numbers.

Although the problem of constructing quadruples of Pythagorean numbers might seem harder and more complicated than the original task of finding Pythagorean triples, it is easier to solve the equation with four unknowns

$$x^2 + y^2 + z^2 = w^2 \tag{1}$$

than to solve the analogous relation with three variables since (1) permits a higher degree of freedom in the selection of variables. Indeed, we can choose two variables x and y almost arbitrarily, and then find z and w from (1).

Our construction of Pythagorean quadruples will make use of the following fact:

In each primitive quadruple (1), *two of the numbers x, y, and z are even.*

(The proof is by contradiction: assuming that x and y are odd, we conclude from (1) that $w^2 - z^2$ is a multiple of 4, and $x^2 + y^2$ is not.)

Using the above fact, let x and y be two arbitrary numbers—one odd and the other even—and find an even number z and odd number w satisfying (1). Represent $x^2 + y^2$ as a product of two different factors:

$$x^2 + y^2 = st \qquad (s > t \geqslant 1).$$

(Note that $t = 1$ if $x^2 + y^2$ is a prime.) Then, from (1), we have $st = (w + z)(w - z)$. Setting $w + z = s$ and $w - z = t$, we obtain

$$z = (s - t)/2 \quad \text{and} \quad w = (s + t)/2. \tag{2}$$

Example 1. Take $x = 1$ and $y = 2$. Then $x^2 + y^2 = 5$; that is, $s = 5$ and $t = 1$. Hence, from (2), we obtain $z = 2$ and $w = 3$. This procedure yields $1^2 + 2^2 + 2^2 = 3^2$.

Example 2. For $x = 3$ and $y = 4$, we have $x^2 + y^2 = 25$ and $s = 25$, $t = 1$. It follows from (2) that $z = 12$, $w = 13$ and $3^2 + 4^2 + 12^2 = 13^2$. This example is of special interest in the sense that the sum of the first two squares is a square, and the sum of the three squares is a square. We may continue this procedure, to find Pythagorean systems containing more than 4 numbers. Using the preceding result, for example, we construct a system $x^2 + y^2 + z^2 + u^2 = v^2$ by solving $13^2 + u^2 = v^2$. Putting $v + u = 169$ and $v - u = 1$, we obtain $u = 84$ and $v = 85$. Thus, $3^2 + 4^2 + 12^2 + 84^2 = 85^2$.

Example 3. Sometimes the discussed method yields more than one Pythagorean quadruple. Take $x = 1$ and $y = 8$. Then $x^2 + y^2 = 65$. To solve the equation $w^2 - z^2 = 65$, factor 65 as $13 \cdot 5$ or $65 \cdot 1$. In this case, we have two systems of equations: $w + z = 13$, $w - z = 5$; and $w + z = 65$, $w - z = 1$. The corresponding solutions are $z = 4$, $w = 9$; and $z = 32$, $w = 33$. Therefore, we obtain two Pythagorean quadruples:

$$1^2 + 8^2 + 4^2 = 9^2 \quad \text{and} \quad 1^2 + 8^2 + 32^2 = 33^2.$$

The terms of the simplest Pythagorean triples are *consecutive* numbers 3, 4, 5. Another equality with similar properties is

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2.$$

To generalize these results, we seek positive integers x that satisfy

$$x^2 + (x+1)^2 + (x+2)^2 + \cdots + (x+n)^2$$
$$= (x+n+1)^2 + (x+n+2)^2 + \cdots + (x+2n)^2, \qquad (3)$$

where n is a natural number. Simplifying (3) gives

$$x^2 - 2n^2x - n^2(2n+1) = 0,$$

the roots of which are

$$x = n(2n+1) \quad \text{and} \quad x = -n.$$

For $n = 1$ and $n = 2$, we have the systems $\{3,4;5\}$ and $\{10,11,12;13,14\}$, respectively. For $n = 3$, we obtain $\{21,22,23,24;25,26,27\}$ since

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2.$$

On the Sphere and Cylinder

Man-Keung Siu, University of Hong Kong, Hong Kong

Archimedes was so fond of his discovery, *that the ratio of the volume of a sphere to that of its circumscribing cylinder is the same as the ratio of their respective surface areas*, that he requested a sphere with its circumscribing cylinder be engraved on his tombstone after he died. Archimedes' result tells us that the surface area of a sphere is the same as the *lateral* surface area of its circumscribing cylinder. Actually, the surface area of any band on the sphere cut off by two parallel planes is the same as that of the projected band on its circumscribing cylinder. This beautiful fact was demonstrated by Arthur Segal in his Classroom Capsule "A Note on the Surface of a Sphere" [TYCMJ 13 (January 1982), 63–64]. Our objective is to view this result in n-dimensional settings where some interesting comparisons and constrasts with the 2- and 3-dimensional cases are possible.

Let $B^n(R) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leqq R^2\}$ be the n-dimensional (closed) ball with radius R and centre O, and let $S^{n-1}(R) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = R^2\}$ be the $(n-1)$-dimensional sphere (sitting in $\mathbb{R}^n$), which is the boundary of $B^n(R)$. We let $v\{B^n(R)\}$ and $v\{S^n(R)\}$ denote the volumes of $B^n(R)$ and $S^n(R)$, respectively. Note, for example, that $v\{B^2(R)\}$ is the area of the "circle" with radius R and $v\{S^1(R)\}$ is its circumference. Also, $v\{B^3(R)\}$ is the volume of the "sphere" with radius R and $v\{S^2(R)\}$ is its surface area. It is reasonable to expect that $v\{B^n(R)\}$ is proportional to R^n, and that $v\{S^{n-1}(R)\}$ is proportional to R^{n-1}. Hence, it is plausible that $v\{S^{n-1}(R)\}$ is equal to $2R$ times $v\{S^{n-2}(R)\}$ times a constant k_n which may depend on n. When $n = 3$, this says that the surface area of a sphere is equal to k_3 times the lateral surface area of its circumscribing cylinder; and from what has been said, k_3 should be 1. This leads one to query: what is k_n in general?