I claim that the simplifying assumption here is made *too early* in the procedure, for to most students it is *completely unmotivated* at this point. In fact, often, the only reason given for the simplifying assumption is that it reduces the amount of work required in determining $y_p''(x)$; unfortunately, this leaves many students wondering why none of the other terms was chosen to be zero instead, for that would just as well make evaluating $y_p''(x)$ easier.

I propose that, rather than making the simplifying assumption at this stage, it is far better to labor through evaluating $y_p''(x)$ in all its goriness. For doing so gives (after some rearrangement)

$$\begin{aligned} f &= y_p'' + py_p' + qy_p \\ &= c_1[y_1'' + py_1' + qy_1] + c_2[y_2'' + py_2' + qy_2] \\ &\quad + [c_1''y_1 + c_1'y_1' + c_2''y_2 + c_2'y_2' + c_1'y_1' + c_2'y_2'] + p[c_1'y_1 + c_2'y_2] \\ &= [c_1''y_1 + c_1'y_1'] + [c_2''y_2 + c_2'y_2'] + p[c_1'y_1 + c_2'y_2] \\ &\quad + [c_1'y_1' + c_2'y_2'] \\ &= \frac{d}{dx}(c_1'y_1) + \frac{d}{dx}(c_1'y_2) + p[c_1'y_1 + c_2'y_2] + [c_1'y_1' + c_2'y_2'] \\ &= \frac{d}{dx}[c_1'y_1 + c_2'y_2] + p[c_1'y_1 + c_2'y_2] + [c_1'y_1' + c_2'y_2']. \end{aligned}$$

Now, at this point, making the assumption that $c_1'y_1 + c_2'y_2 = 0$ is well motivated, for it eliminates almost all the terms in the last line *and* yields the system

$$\begin{cases} c_1'y_1 + c_2'y_2 = 0 \\ c_1'y_1' + c_2'y_2' = f \end{cases}$$

in one fell swoop!

Teaching the Laplace Transform Using Diagrams

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In this capsule, we present an approach to evaluating the Laplace transform and its inverse using commutative diagrams. The value of this technique is twofold. First, it presents a visual approach to a symbolic process. Second, it introduces the concept of commutative diagrams, which embody the important idea of distinct processes producing identical results.

The *Laplace transform* of a function $f(t)$ $(t > 0)$ is defined by

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

for all values s for which the integral is defined. We write $f(t) = \mathscr{L}^{-1}\{F(s)\}$ and call $f(t)$ the *inverse Laplace transform* of $F(s)$. If a is a real constant and n is a positive integer, we have

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \quad (s > a) \qquad \mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad (s > 0)$$
$$\mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2} \quad (s > 0) \qquad \mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2} \quad (s > 0).$$

The following theorem can be found in standard differential equations texts.

Theorem 1. *Let $f(t)$ be a function with Laplace transform $F(s)$. Then the Laplace transform of the function $\int_0^t f(\tau)\,d\tau$ is $(1/s)F(s)$.*

This theorem is illustrated by the following diagram.

$$\begin{array}{ccc} f(t) & & F(s) \\ & & \\ \displaystyle\int_0^t f(\tau)\,d\tau & & \dfrac{1}{s}F(s) \end{array}$$

This diagram indicates that integrating $f(t)$ corresponds to dividing its Laplace transform by s.

Calculation of Laplace transforms and their inverses may be illustrated in a similar way.

Example. Find the inverse Laplace transform of $1/(s^2(s-1))$.

Solution. We see that $1/(s^2(s-1))$ can be obtained from $1/(s-1)$, whose inverse Laplace transform is e^t, by dividing by s twice. We draw the corresponding diagram, filling in the appropriate inverse transforms. The answer is in the lower left corner.

$$\begin{array}{rcl} e^t & & \dfrac{1}{s-1} \\ & & \\ \displaystyle\int_0^t e^\tau\,d\tau = e^t - 1 & & \dfrac{1}{s(s-1)} \\ & & \\ \displaystyle\int_0^t (e^\tau - 1)\,d\tau = \boxed{e^t - t - 1} & & \dfrac{1}{s^2(s-1)} \end{array}$$

Other properties can be illustrated in the same manner.

Theorem 2. *Let $f(t)$ be a function with Laplace transform $F(s)$. Then the Laplace transform of the function $tf(t)$ is $-F'(s)$.*

$$\begin{array}{ccc} f(t) & & F(s) \\ & & \\ tf(t) & & -F'(s) \end{array}$$

This diagram indicates that multiplying $f(t)$ *by* t corresponds to differentiating its Laplace transform and multiplying by -1.

Example. Find the inverse Laplace transform $f(t)$ of $\ln(s^2/(s^2+1))$.

Solution. We construct the following diagram

$$
\begin{tikzcd}
f(t) \arrow[r, dash] \arrow[d, dash] & \ln\dfrac{s^2}{(s^2+1)} = 2\ln s - \ln(s^2+1) \arrow[d, dash] \\
tf(t) = -2 + 2\cos t \arrow[r, dash] & -\dfrac{2}{s} + \dfrac{2s}{s^2+1}
\end{tikzcd}
$$

from which we see that $tf(t) = -2 + 2\cos t$ and so $\boxed{f(t) = \dfrac{2\cos t - 2}{t}}$.

Two other important properties of the Laplace transform can be illustrated by the self-explanatory diagrams

$$
\begin{tikzcd}
f(t) \arrow[r, dash] \arrow[d, dash] & F(s) \arrow[d, dash] \\
e^{at}f(t) \arrow[r, dash] & F(s-a)
\end{tikzcd}
\qquad
\begin{tikzcd}
f(t) \arrow[r, dash] \arrow[d, dash] & F(s) \arrow[d, dash] \\
u_a(t)f(t-a) \arrow[r, dash] & e^{-as}F(s)
\end{tikzcd}
$$

where $u_a(t) = \begin{cases} 1 & t > a \\ 0 & t < a. \end{cases}$

Example. Find the Laplace transform of $\int_0^t ue^u \cos u\, du$.

Solution. We obtain the answer in the lower right hand corner of the following diagram.

$$
\begin{tikzcd}
\cos t \arrow[r, dash] \arrow[d, dash] & \dfrac{s}{s^2+1} \arrow[d, dash] \\
t\cos t \arrow[r, dash] \arrow[d, dash] & \dfrac{s^2-1}{(s^2+1)^2} \arrow[d, dash] \\
te^t \cos t \arrow[r, dash] \arrow[d, dash] & \dfrac{(s-1)^2-1}{\left[(s-1)^2+1\right]^2} = \dfrac{s^2-2s}{(s^2-2s+2)^2} \arrow[d, dash] \\
\displaystyle\int_0^t ue^u \cos u\, du \arrow[r, dash] & \boxed{\dfrac{s-2}{(s^2-2s+2)^2}}
\end{tikzcd}
$$

We have found this diagrammatic approach very effective in enforcing the concept of the Laplace transform and its properties, while also providing a visual way of keeping track of the processes used in calculations.

Reference

R. V. Churchill, *Operational Mathematics*, 3rd ed., McGraw Hill, New York, 1972.

A Serendipitous Application of the Pythagorean Triplets

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On occasion, a purely pedagogical consideration leads to an interesting mathematical result. To determine whether students had a good grasp of the process of factoring monic quadratic polynomials, I asked them to factor pairs of the form

$$(x^2+px+q,\ x^2+px-q), \qquad p,q\neq 0 \tag{1}$$

where each polynomial has integer zeros. Examples are:

(i) $(x^2+5x+6,\ x^2+5x-6)$

(ii) $(x^2+13x+30,\ x^2+13x-30)$

(iii) $(x^2+17x+60,\ x^2+17x-60)$.

The natural question arose as to whether it is possible to produce all such pairs of polynomials. As we will see, the answer is yes.

We begin with the observation that once a pair of the desired polynomials is known, an infinite number of pairs can be generated from it; for if the polynomials in (1) have integer zeros, so do $(x^2+tpx+t^2q,\ x^2+tpx-t^2q)$ for each integer t. We are merely producing polynomials whose zeros have been multiplied by t. This motivates the following definitions:

Definitions. A pair of polynomials of the form described in (1) is an *integer quadratic polynomial pair* if each polynomial has integer zeros. If the polynomials in (1) satisfy the additional requirement that $(p,q)=1$, then the polynomials will be called a *representative pair*.

It is readily seen that if $t|(p,q)$ and $(x^2+px+q,\ x^2+px-q)$ is an integer quadratic polynomial pair, then $(x^2+(p/t)x+q/t^2,\ x^2+(p/t)x-q/t^2)$ is also such a pair. It follows that each integer quadratic polynomial pair can be derived from a representative pair and therefore, it suffices to produce all representative pairs. To this end, we prove the following theorem.