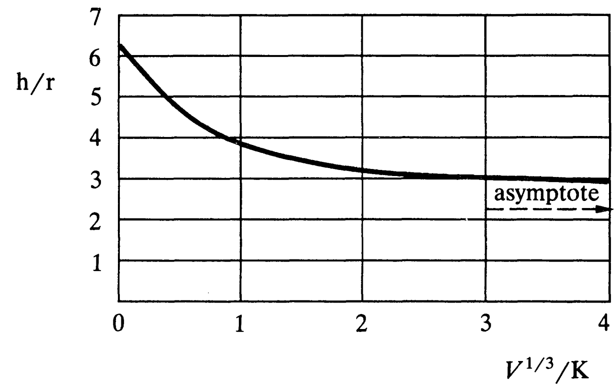

Figure 1

how the can-making machine actually worked? Are we designing to a given volume? Or to a given weight of contents? How are the cans stacked for transport? How much does it cost to be slightly off the optimal proportions? Is this offset by any other saving?

There is plenty of scope here for developing a true appreciation of how mathematics contributes to technology.

The Curious 1/3

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The United States Postal Service will accept for delivery only packages that conform to this rule: the length plus the girth must not exceed 108 inches. This rule generates a number of maximization problems in calculus texts [**1**, p. 260], [**2**, p. 240], [**3**, p. 220]. The answers to these problems contain an interesting surprise.

To generalize slightly we may replace 108 by any positive constant c. The typical textbook problem asks for the maximum volume of a right cylindrical package when the cross section of the package is a square or when it is a circle. The length of the box of maximum volume is, in both cases, $c/3$.

In a recent class I asked the students to try using equilateral triangles for the cross section. The length of the box of maximum volume was still $c/3$. The reader might want to try other shapes, perhaps an isosceles right triangle.

Is this just a strange coincidence? Or is the length of the box of maximum volume actually independent of the shape of the cross section?

Suppose we decide on a (reasonable) shape for the cross section of the box. Consider one example of that shape with perimeter P and area A. We may use a "magnification factor" x to describe all similar shapes, which will have perimeter Px and area Ax^2. (It should be easy to convince students of the existence of such a magnification factor, at least in the case of figures that can be decomposed into a finite number of triangles. For more complicated figures and more advanced students an argument using line integrals for the perimeter and area should be convincing.)

Let z be the length of the box. We seek the maximum volume $V = Ax^2z$ subject to the restriction $Px + z = c$. This constraint can be solved for x to yield $V = (A/P^2)z(c-z)^2$ for $0 \le z \le c$.

The maximizing z is clearly independent of the shape and always $c/3$. However, the maximum volume $(4A/P^2)(c/3)^3$ does depend on the shape through the ratio $4A/P^2$. Indeed, the appearance of the isoperimetric ratio $4A/P^2$ in the solution clarifies the situation and opens up a whole new avenue for classroom discussion.

Note that two-thirds of c is left for the perimeter of the cross section so that maximum dimension of the cross section cannot exceed $c/3$, one-half of the perimeter. The largest dimension of the box is indeed what we have called the length.

One last question. What is the "correct" generalization to other dimensions?

References

1. Ross L. Finney and George B. Thomas, *Calculus*, Addison-Wesley, Reading, MA, 1990.
2. Phillip Gillett, *Calculus and Analytic Geometry*, 3rd Edition, Heath, Lexington, MA, 1988.
3. Al Shenk, *Calculus and Analytic Geometry*, 3rd Edition, Scott, Foresman, Glenview, IL, 1984.
4. W. H. Bussey, Maximum parcels under the new parcel post law, *American Mathematical Monthly* 20 (1913) 58–59, reprinted in *Selected Papers on Calculus*, MAA, 1969, pp. 232–233.

What is the Biggest Rectangle You Can Put Inside a Given Triangle?

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The purpose of this note is to put together an instructive package of pleasant established theorems dealing with certain maximum rectangles inside triangles and several other results of this flavor.

The following familiar old calculus book problem is dealt with in [**1**], [**3**], [**5**], [**6**], and [**7**] in one form or another: *Given a triangle of altitude a and base b, find the dimensions of the rectangle of maximum area that can be inscribed in this triangle with one side along the base.* (In many old books the triangle is a *right* triangle.)

We first remind ourselves of a solution of this problem. See Figure 1, where the two essentially different cases are shown; in (i) but not in (ii), the vertex C is "above" the base.

Starting with (i) in Figure 1, we seek the maximum of the product xy, where, from similar triangles, we have $y = (b/a)(a - x)$. Thus, $xy = (b/a)(x)(a - x)$, and we need to find x so that this is maximized. We need not use a derivative; we can complete the square to observe that $x(a - x) = (a/2)^2 - \{x - (a/2)\}^2$, and this is a

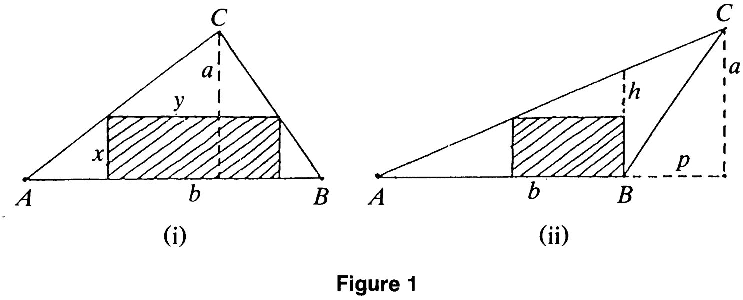

Figure 1