where the summation on the right is taken over all the 2^n combinations of the $\pm$ signs.

For a generalization in another direction, note that (1) can be rewritten as

$$|z_1|^2+|z_2|^2=\left|(z_1-z_2)/\sqrt{2}\right|^2+\left|(z_1+z_2)/\sqrt{2}\right|^2.$$

In the real plane, the transformation $x'=(x-y)/\sqrt{2}$, $y'=(x+y)/\sqrt{2}$, represents a rotation of the coordinate axes by $45°$ and preserves all distances, i.e., $\sqrt{x^2+y^2}=\sqrt{x'^2+y'^2}$. For the case here, the value of $|z_1|^2+|z_2|^2$ is preserved under an orthogonal transformation. More generally (as is known), if $z_1, z_2, \dots, z_n$ are complex numbers (or vectors in space) and we make the transformation $Z'=MZ$ where M is an arbitrary real orthogonal matrix and the transpose matrices of Z and Z' are

$$Z^T=(z_1, z_2, \dots, z_n) \quad \text{and} \quad Z'^T=(z_1', z_2', \dots, z_n'),$$

then

$$\sum |z_i'|^2=\sum |z_i|^2$$

and its proof is quite direct:

$$\sum |z_i'|^2=\sum z_i'\bar{z}_i'=Z'^T\bar{Z}'=(MZ)^T(\overline{MZ})=Z^TM^TM\bar{Z}=Z^T\bar{Z}=\sum |z_i|^2$$

(since M is orthogonal $M^TM=I$). The proof for vectors is the same except that the multiplication of the two vector matrices Z^T and $\bar{Z}$ is via the scalar dot product.

More generally, the matrix M can be replaced by a complex matrix U if it is unitary, i.e., $U^T\bar{U}=I$. Finally, the identity (3) can be generalized by replacing the z_i's by z_i''s and then letting $Z'=UZ$. For a simple example in (2), let

$$z_1'=iz_1\cos\theta+iz_2\sin\theta, \qquad z_2'=z_1\sin\theta-z_2\cos\theta, \qquad z_3'=z_3$$

where θ is an arbitrary real angle.

A Generalization of $\lim\limits_{n\to\infty}\sqrt[n]{n!}/n=e^{-1}$

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In "Alternate Approaches to Two Familiar Results" [CMJ 15 (November 1984) 422–423], we gave an elementary proof of the familiar result

$$\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}=e^{-1}. \tag{1}$$

The following generalization of (1) states that for any nonnegative integer s:

$$\lim_{n\to\infty}\frac{\left(1^{(1^s)}\cdot 2^{(2^s)}\cdots n^{(n^s)}\right)^{1/n^{s+1}}}{n^{1/(s+1)}}=e^{-1/(s+1)^2}. \tag{2}$$

Thus, for example, putting $s = 0$ gives (1). And letting $s = 1$, we get

$$\lim_{n\to\infty} \frac{\left(1^1 \cdot 2^2 \cdots n^n\right)^{1/n^2}}{n^{1/2}} = e^{-1/4}, \tag{3}$$

which can also be obtained from Glaisher's formula for approximating $1^1 \cdot 2^2 \cdots n^n$. (See, for instance, p. 55 of Konrad Knopp's *Theory and Application of Infinite Series*, Blackie & Son, 1963.)

In this note, we illustrate how the mean value theorem can be used to prove (2). It suffices to consider the case $s = 2$:

$$\lim_{n\to\infty} \frac{\left(1^{(1^2)} \cdot 2^{(2^2)} \cdots n^{(n^2)}\right)^{1/n^3}}{n^{1/3}} = e^{-1/9}. \tag{4}$$

Using the MVT with $f(x) = \dfrac{x^3 \ln x}{3} - \dfrac{x^3}{9}$, we have

$$\left[\frac{(k+1)^3\ln(k+1)}{3} - \frac{(k+1)^3}{9}\right] - \left[\frac{k^3\ln k}{3} - \frac{k^3}{9}\right] = c^2 \ln c$$

for some $c \in (k, k+1)$. It follows that

$$\begin{aligned} k^2 \ln k &< \left[\frac{(k+1)^3\ln(k+1)}{3} - \frac{(k+1)^3}{9}\right] - \left[\frac{k^3\ln k}{3} - \frac{k^3}{9}\right] \\ &< (k+1)^2\ln(k+1). \end{aligned} \tag{5}$$

Rewriting (5) for each value of k ($k = n, n-1, \ldots, 1$) and adding, we get

$$\sum_{k=1}^{n} k^2 \ln k < \frac{(n+1)^3\ln(n+1)}{3} - \frac{(n+1)^3}{9} + \frac{1}{9} < \sum_{k=2}^{n+1} k^2 \ln k.$$

This can be written as

$$\begin{aligned} &\frac{(n+1)^3\ln(n+1)}{3} - (n+1)^2\ln(n+1) - \frac{(n+1)^3}{9} + \frac{1}{9} \\ &\qquad < \sum_{k=1}^{n} k^2 \ln k < \frac{(n+1)^3\ln(n+1)}{3} - \frac{(n+1)^3}{9} + \frac{1}{9}, \end{aligned}$$

or as

$$\begin{aligned} &\ln\left[(n+1)^{[(n+1)^3/3 - (n+1)^2]} \cdot e^{[1-(n+1)^3]/9}\right] \\ &\qquad < \ln\left[\prod_{k=1}^{n} k^{(k^2)}\right] < \ln\left[(n+1)^{(n+1)^3/3} \cdot e^{[1-(n+1)^3]/9}\right]. \end{aligned}$$

Thus, we have

$$\begin{aligned} &(n+1)^{[(n+1)^3/3 - (n+1)^2]} \cdot e^{[1-(n+1)^3]/9} < 1^{(1^2)} \cdot 2^{(2^2)} \cdots n^{(n^2)} \\ &\qquad < (n+1)^{(n+1)^3/3} \cdot e^{[1-(n+1)^3]/9}. \end{aligned}$$

It follows that

$$\frac{(n+1)^{(n^3-3n-2)/3n^3}\cdot e^{[1-(n+1)^3]/9n^3}}{n^{1/3}} < \frac{\left[1^{(1^2)}\cdot 2^{(2^2)}\cdots n^{(n^2)}\right]^{1/n^3}}{n^{1/3}} < \frac{(n+1)^{(n^3+3n^2+3n+1)/3n^3}\cdot e^{[1-(n+1)^3]/9n^3}}{n^{1/3}},$$

or

$$\frac{(n+1)^{1/3}}{n^{1/3}}\cdot(n+1)^{(-3n-2)/3n^3}\cdot e^{[1-(n+1)^3]/9n^3} < \frac{\left[1^{(1^2)}\cdot 2^{(2^2)}\cdots n^{(n^2)}\right]^{1/n^3}}{n^{1/3}} < \frac{(n+1)^{1/3}}{n^{1/3}}\cdot(n+1)^{(3n^2+3n+1)/3n^3}\cdot e^{[1-(n+1)^3]/9n^3}.$$

Now, note that as $n \to \infty$:

$$\frac{(n+1)^{1/3}}{n^{1/3}} \to 1, \qquad (n+1)^{(-3n-2)/3n^3} \to 1,$$

$$(n+1)^{(3n^2+3n+1)/3n^3} \to 1, \qquad e^{[1-(n+1)^3]/9n^3} \to e^{-1/9}.$$

Hence,

$$\lim_{n\to\infty} \frac{\left(1^{(1^2)}\cdot 2^{(2^2)}\cdots n^{(n^2)}\right)^{1/n^3}}{n^{1/3}} = e^{-1/9}.$$

In the same manner, we can begin with

$$f(x) = \frac{x^{s+1}\ln x}{s+1} - \frac{x^{s+1}}{(s+1)^2}$$

to establish (2) in general.

Notational Collisions

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"Interpreting a symbol is to associate it with some concept or mental image to assimilate it to human consciousness."

The Mathematical Experience, Davis and Hersh

Mathematics is often eulogized for its compact and concise notational systems and for its use of symbols to represent (possibly quite complex) objects, constructions,