6. Konrad J. Heuvers, Symmetric matrices with prescribed eigenvalues and eigenvectors, *Mathematics Magazine*, 55 (1982) 106–111.
7. John G. Kemeny, How computers have changed the way I teach, *Academic Computing*, 2 (1988) 44–46, 59–61.
8. J. M. Ortega, Letter to the editor, *American Mathematical Monthly*, 92 (1985) 526.
9. J. C. Renaud, Matrices with integer entries and integer eigenvalues, *American Mathematical Monthly*, 90 (1983) 202–203.
10. Gilbert Strang, *Linear Algebra and Its Applications*, third ed., Academic Press, New York, 1988.
11. Alan Tucker, *A Unified Introduction to Linear Algebra: Methods, Models and Theory*, Macmillan, New York, 1988.
12. MATLAB™ software by The MathWorks, Inc., Natick, MA.

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## Rotation Matrices in the Plane without Trigonometry

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For a fixed angle $\theta$, the rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

induces a rotation of the plane by the angle $\theta$ about the origin. That is, for any column vector ($2 \times 1$ matrix) $\mathbf{v}$ in $\mathbb{R}^2$, the product of this matrix with $\mathbf{v}$ is the rotation of $\mathbf{v}$ by the angle $\theta$ about the origin. Ordinarily, this rotation property is established by a trigonometric argument that relies on the angle addition formulas for the sine and cosine functions. For example see [C. H. Edwards, Jr. and D. E. Penney, *Calculus and Analytic Geometry*, 3rd ed., Prentice Hall, Englewood Cliffs, NJ, 1990, p. 491]. In this capsule we reverse the procedure. We provide a simple and direct justification of the rotation property of rotation matrices that is virtually free of trigonometric arguments, and we then use our results to derive these trigonometric identities.

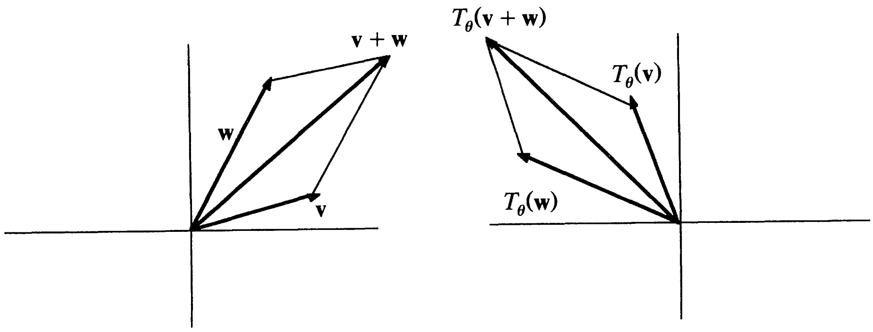
**The rotation operator.** We begin with the mapping on $\mathbb{R}^2$ that rotates a vector by a fixed angle $\theta$ about the origin so that the rotation is counter-clockwise if $\theta$ is positive, and the origin is fixed under the rotation. Let us denote this mapping by $T_\theta$. We argue that $T_\theta$ is linear, and then show that its matrix representation with respect to the standard ordered basis for $\mathbb{R}^2$ is the rotation matrix given above. This establishes the rotation property for this matrix. Finally, we use the rotation property to derive the desired trigonometric identities.

In what follows we list certain basic geometric properties about $T_\theta$ that are inherent in the idea of rotation:

1. The rotation of a vector by the angle $\alpha$ followed by the rotation by the angle $\beta$ is equivalent to the rotation of the vector by $\alpha+\beta$. Symbolically: $T_\beta T_\alpha = T_{\alpha+\beta} = T_{\beta+\alpha} = T_\alpha T_\beta$.
2. The angle of intersection of two vectors is preserved under any rotation.
3. The length of a vector is preserved under any rotation.

To show that $T_\theta$ is linear, consider any two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$. We may think of $\mathbf{v}$ and $\mathbf{w}$ as adjacent sides of a parallelogram intersecting at the origin. Then $\mathbf{v}+\mathbf{w}$ is the diagonal of this parallelogram with the origin as an endpoint. The rotated vectors $T_\theta(\mathbf{v})$ and $T_\theta(\mathbf{w})$ are now adjacent sides of a new (rotated) parallelogram. See the figures below. One can now argue that the rotated diagonal,

$T_\theta(\mathbf{v}+\mathbf{w})$, is a diagonal of this new parallelogram. The argument can be based on purely geometric grounds using (2) and (3) of the basic geometric properties of rotations listed above and facts about congruent figures. We omit the details.




Consequently, the rotation of the diagonal of the old parallelogram is the same as the diagonal of the new parallelogram. That is,

$$T_\theta(\mathbf{v}+\mathbf{w}) = T_\theta(\mathbf{v}) + T_\theta(\mathbf{w}).$$

Trivially, if $\mathbf{v}$ or $\mathbf{w}$ or both are zero, then we also have this additive property of $T_\theta$. So we have the additive property of $T_\theta$ for any vectors $\mathbf{v}$ and $\mathbf{w}$. The multiplicative property of $T_\theta$ can also be justified by a geometric argument using properties (2) and (3). We omit the details. With these two properties we have that $T_\theta$ is linear.

**The rotation matrix.** Let

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

be the standard vectors of $\mathbb{R}^2$ and let $A_\theta$ be the matrix representation of $T_\theta$ with respect to the ordered basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. Recall that the columns of $A_\theta$ are $T_\theta(\mathbf{e}_1)$ and $T_\theta(\mathbf{e}_2)$, respectively. The elementary theory of linear algebra tells us that $T_\theta(\mathbf{v}) = A_\theta \mathbf{v}$ for any vector $\mathbf{v}$ in $\mathbb{R}^2$. Our task, therefore, is to show that $A_\theta$ is the rotation matrix mentioned in the introduction.

We begin with the special case that $\theta = \pi/2$. Clearly, $T_{\pi/2}(\mathbf{e}_1) = \mathbf{e}_2$ and $T_{\pi/2}(\mathbf{e}_2) = -\mathbf{e}_1$. Hence

$$A_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Now let $\theta$ be any angle. Then $T_\theta(\mathbf{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ by the definitions of cosine and sine. To find $T_\theta(\mathbf{e}_2)$ observe that $\mathbf{e}_2$ is the rotation of $\mathbf{e}_1$ by $\pi/2$, and hence

$$\begin{aligned} T_\theta(\mathbf{e}_2) &= T_\theta T_{\pi/2}(\mathbf{e}_1) \\ &= T_{\pi/2} T_\theta(\mathbf{e}_1) \quad \text{(by property 1 of rotations)} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\ &= \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}. \end{aligned}$$

It follows, therefore, that

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

which is the rotation matrix mentioned in the introduction. So we have established the rotation property of rotation matrices.

**Trigonometric consequences.** Now that we have the desired rotation property we can use it to derive the familiar angle addition formulas for the sine and cosine functions.

**Theorem.** *$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.*

*Proof.* Let $\mathbf{v} = \left[\begin{smallmatrix} \cos\beta \\ \sin\beta \end{smallmatrix}\right]$. Then $A_\alpha\mathbf{v}$ is the result of rotating $\mathbf{v}$ by $\alpha$, and therefore

$$A_\alpha\mathbf{v} = \begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix}.$$

Furthermore,

$$\begin{aligned} A_\alpha\mathbf{v} &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{bmatrix}.$$

By comparing the corresponding components of the vectors in this last equation we have the desired results.

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## A Geometric Interpretation of the Columns of the (Pseudo)Inverse of A

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This capsule describes how the columns of the (pseudo)inverse of a matrix $A$ can be used to provide useful geometric information about the rows of $A$. Specifically, it shows how the $i$th column of the (pseudo)inverse of $A$ can be used to project the $i$th row of $A$ on the span of the other rows (see Figure 1). We begin with an elementary proof of the important special case for which the row space of $A$ spans all of Euclidean $n$-space $E_n$.