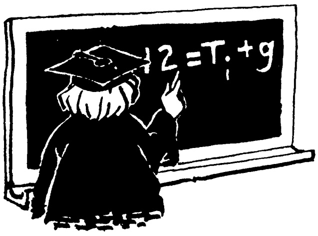


CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

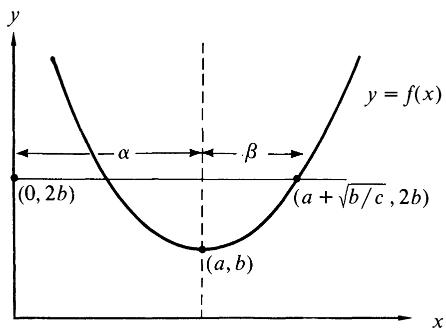
Readers are invited to submit material for consideration to:

Warren Page
Department of Mathematics
New York City Technical College
300 Jay Street
Brooklyn, N.Y. 11201

Complex Roots Made Visible

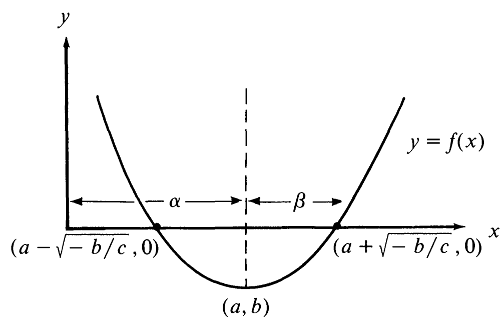
Alec Norton and Benjamin Lotto, University of California, Berkeley, CA

The quadratic formula gives a complete *algebraic* solution to the problem of finding the roots of a real quadratic equation $f(x) = 0$, but the corresponding geometric interpretation seems to be well known only in the case of real roots (when the roots are displayed as the intersection of the graph of f with the x-axis). Thus, it may come as a pleasant surprise that there is also a simple, natural way to geometrically read off the complex roots of $f(x) = 0$.



$f(x) = 0$ has roots $\alpha \pm i\beta = a \pm i\sqrt{b/c}$.

Figure 1.



$f(x) = 0$ has roots $\alpha \pm \beta = a \pm \sqrt{-b/c}$.

Figure 2.

Suppose $f(x)$ is concave upward and has roots $\alpha \pm i\beta$. Then α and β are as shown in Figure 1. Specifically, α is the abscissa of the vertex and β is half the length of the chord determined by the horizontal line $y = 2b$, where b is the ordinate of the vertex. The case for $f(x)$ concave down is analogous.

To verify the interpretation of α and β in Figure 1, let $f(x) = c(x - a)^2 + b$ for $b, c > 0$. Then f has roots $a \pm i\sqrt{b/c}$, and $\alpha = a$ and $\beta = \sqrt{b/c}$ are precisely the asserted lengths.

This interpretation seems natural because in the case of real roots the picture (Figure 2) is the same, except that now the horizontal line is merely the x-axis itself; the roots are $\alpha \pm \beta$ instead of $\alpha \pm i\beta$, where $\beta = \sqrt{|b/c|}$.

We can now see both of these pictures unified in a simple bifurcation process: imagine a parabola (say, $b < 0 < c$) moving uniformly in the positive y-direction, and observe the behavior of the roots in the complex plane. As b increases toward 0, we see that $\beta = \sqrt{|b/c|}$ decreases toward 0. Thus, as long as the parabola meets the x-axis, the roots are real, symmetric about $x = a$, and are converging toward each other (Figure 3a). At the bifurcation point, $b = 0$ and the roots are coincident (Figure 3b). As b increases above 0, β increases and the two roots diverge in the imaginary direction (Figure 3c) at the same rate as their previous motion in Figure 3a.

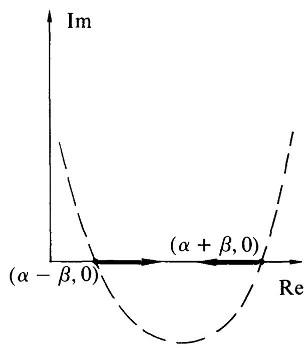


Figure 3a.

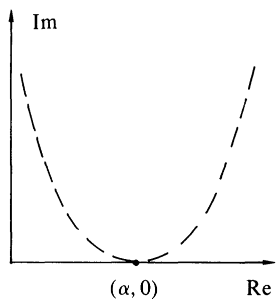


Figure 3b.

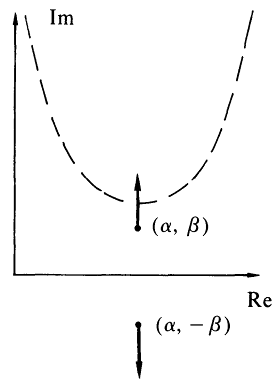


Figure 3c.

Another Look at $x^{1/x}$

Norman Schaumberger, Bronx Community College, Bronx, NY

Here is a nice way for students to apply and combine two important results. First, we use the inequality between the arithmetic and geometric means to prove that the sequence $\sqrt[n]{n}$ tends to 1 as $n \to \infty$. Then the Mean Value theorem for derivatives will be used to show that the maximum value of $x^{1/x}$ occurs at $x = e$.

According to the arithmetic-geometric mean inequality,

$$\sqrt[n]{a_1 a_2 \cdots a_n} < \frac{a_1 + a_2 + \cdots + a_n}{n}$$

when the positive numbers $a_1, a_2, \ldots, a_n$ are not all equal. Suppose we set $a_1 = a_2 = \cdots = a_{n-1} = 1$ and let $a_n = \sqrt{n}$. Then, since $\sqrt[n]{\sqrt{n}} > 1$, we get