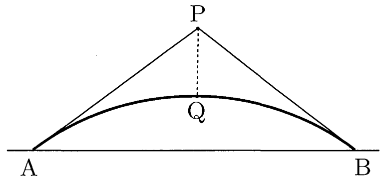

Figure 1

Consider an arbitrary partition of $[a, b]$. Summing the inequalities (1) as $[u, v]$ ranges over the subintervals of the partition, we find that L_a^b is bounded below by the lower sum for the function $\sqrt{1+f'^2}$ and is bounded above by the upper sum for this function. Since the partition was arbitrary, it follows that L_a^b lies between *all* lower sums and *all* upper sums for $\sqrt{1+f'^2}$. But the unique number with this property is the integral. Therefore,

$$L_a^b = \int_a^b \sqrt{1+f'^2}\, dx. \tag{2}$$

It is worth pointing out that in the presence of additivity of arc length, the integral formula (2) is in fact *equivalent* to Axiom L. We have just derived the formula from the axiom. Conversely, under the hypotheses of Axiom L the formula immediately implies the axiom's conclusion.

It is also interesting to observe how simply Axiom L implies the known fact that the perimeter of a circumscribed polygon is an upper bound for the circumference of a circle. Consider the arc of the circle between two consecutive points of tangency, A and B. Position the coordinate axes with A and B on the x-axis, let P be the vertex of the polygon, and let Q be the midpoint of the arc, as in Figure 1. By axiom L, AP is longer than AQ and PB is longer than QB. Hence AP + PB is longer than the arc AQB, which implies the result.

References

1. P. D. Barry, On arc length, *College Mathematics Journal* 28:5 (1997) 338–347.
2. J. L. Coolidge, The lengths of curves, *American Mathematical Monthly* 60 (1953) 89–93.
3. Leonard Gillman, An axiomatic approach to the integral, *American Mathematical Monthly* 100:1 (1993) 16–25.
4. Howard Levi, *Polynomials, Power Series, and Calculus*, Van Nostrand, New York, 1968.

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The Buckled Rail: Three Formulations

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Calculating the shape of a steel rail that has elongated by expansion is likely to yield some surprising results for students. Let's consider a rail that is one mile long and is hinged at each end. Now suppose that we extend its length by one inch while

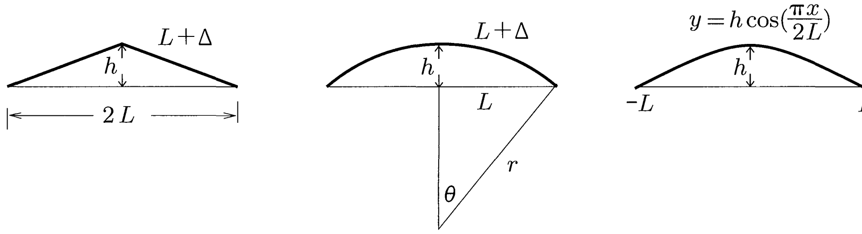

Figure 1. *Left to right:* The buckled shape is a triangle, an arc of a circle, or sinusoidal.

keeping the ends fixed, so that the rail buckles upward. Think what it might look like: Is the center raised just enough to slip a piece of paper under? Is it tall enough to walk under? Or is it so high that you could drive a tractor-trailer through?

In exploring this situation, we shall make three separate assumptions about the buckled shape, to yield three separate problems whose solutions have their own distinct surprises. The assumptions are:

1. The rail is hinged in the middle, so the buckled shape is a triangle.
2. The shape is an arc of a circle.
3. The shape is sinusoidal.

These alternatives are shown in Figure 1. The techniques for solving this progression of models move from the simple to the rather sophisticated.

Hinged rail. The height of the center is easily found, using the Pythagorean theorem, to be $h = \sqrt{(L+\Delta)^2 - L^2} = \sqrt{2L\Delta}[1 + O(\Delta/L)]$. For $L = 0.5$ mile $= 31{,}680$ inches and $\Delta = 0.5$ inch, we find $h = 178$ inches or 14.8 feet. A rather dramatic facet of this result is that with Δ fixed, $h \to \infty$ as $L \to \infty$.

Circular arc. Many students find the second model much more difficult than the first, because they have never had to formulate a problem whose solution requires solving a transcendental equation. From the relations

$$\theta = \frac{L+\Delta}{r} \quad \text{and} \quad \sin\theta = \frac{L}{r},$$

we obtain

$$\sin\theta = \frac{L}{L+\Delta}\theta. \tag{1}$$

We cannot get a closed-form solution for θ from (1). Moreover, because $L/(L+\Delta)$ is so close to 1 for our values, it is a little difficult to find a numerical solution to the equation. If we note that θ is very small, we may approximate $\sin\theta \approx \theta - \theta^3/3!$ and find

$$\theta \approx \left(\frac{6\Delta}{L+\Delta}\right)^{1/2}. \tag{2}$$

The central diagram in Figure 1 shows us that

$$h = \frac{L(1-\cos\theta)}{\sin\theta}. \tag{3}$$

From (3) and the approximations for sine and cosine, we find

$$h \approx \frac{L}{2}\theta \approx \frac{L}{2}\sqrt{\frac{6\Delta}{L+\Delta}}. \tag{4}$$

For the values of L and Δ in our example, we find $h \approx 154$ inches.

Sinusoidal shape. Within the assumptions of elementary beam theory, the true shape of a buckled beam is a sine function (or a cosine if the origin is located at the center). [See S. Timoshenko, *Strength of Materials*, part I, 3rd ed., Van Nostrand, New York, 1955, pp. 258 ff.] Therefore, this model is the most realistic of the three. It is also the most difficult for students to formulate and solve. We will use the formula for arc length $\int_{-L}^{L}\sqrt{1+y'^2}\,dx = 2L+2\Delta$.

Taking $y = h\cos[\pi x/(2L)]$, we obtain an equation for h:

$$\int_{-L}^{L}\left[1+\left(\frac{h\pi}{2L}\right)^2\sin^2\left(\frac{\pi}{2L}x\right)\right]^{1/2}dx = 2L+2\Delta. \tag{5}$$

Setting $x = 2Lt/\pi$ and $k = [h\pi/(2L)]^2$, then using the symmetry of the integral, we obtain

$$\int_0^{\pi/2}\left(1+k\sin^2 t\right)^{1/2}dt = \frac{\pi(L+\Delta)}{2L}. \tag{6}$$

To solve (6) for k, we expand the integrand in a power series in k and integrate term by term:

$$\left(1+k\sin^2 t\right)^{1/2} = 1+\frac{1}{2}k\sin^2 t + \mathrm{O}(k^2).$$

When we insert this series into (6) and drop terms of $\mathrm{O}(k^2)$, we find $k \approx \dfrac{4\Delta}{L}$. But $k = \dfrac{\pi^2}{(2L)^2}h^2$; therefore,

$$h \approx \frac{4}{\pi}\sqrt{L\Delta}. \tag{7}$$

For this shape, we find $h \approx 160$ in.

Before my students try to solve these problems, I ask them to guess which model will yield the highest value for the center of the rail. Most can immediately see that the triangle will be the highest, but there is usually disagreement about the other two. Generally, the cosine shape gets the most votes for second place. For small values of Δ/L, the results are summarized in the following table.

Model type	*Approximate center height*
Triangle	$\sqrt{2}\sqrt{L\Delta}$
Cosine	$\frac{4}{\pi}\sqrt{L\Delta}$
Circular	$\sqrt{\frac{3}{2}}\sqrt{L\Delta}$

Since $\sqrt{2} > 4/\pi > \sqrt{3/2}$, the conjecture about the heights is validated for very small values of Δ/L.

We have seen three different models for the buckling of the rail, each using a different mathematics for its formulation. For a given length and extension, each may be solved using a computer algebra system. The dimensionless form of the solution is $h/L = \sqrt{\Delta/L}\, g(\Delta/L)$ where g is an analytic function of its argument. We have, therefore, given the leading term of the series for g. In many instances, a formula reveals relations that are not readily apparent in a numerical solution. In these three problems, for example, we see that the buckled height is proportional to the geometric mean of L and Δ.

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Who Cares If $X^2 + 1 = 0$ Has a Solution?

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> The shortest path between two truths in the real domain passes through the complex domain.—*Jacques Hadamard*

Most mathematics textbooks introduce complex numbers as a means of solving equations that obviously have no real solutions. A typical introduction goes something like this:

> The equation $x^2 + 1 = 0$ has no real solution because there is no real number x that can be squared to produce -1. To solve such an equation, mathematicians created an expanded system of numbers using the imaginary unit i, defined as $i = \sqrt{-1}$.

A student may well ask: Why solve this equation in the first place? And in any case, who cares if it has a solution?

These are legitimate questions. One would expect a practical or intuitive justification for introducing such a novel idea. After all, there are direct and intuitive motivations for introducing other aspects of our number system. The natural numbers are used for counting, negative numbers may be used to describe debt, rational numbers help us describe such natural concepts as "half a quart of milk," and irrational numbers are needed for representing certain distances in the plane. On the other hand, there is no easy application of complex numbers that serves to motivate their use at the usual introductory level. Moreover, by the time students are sophisticated enough to understand the applications of complex numbers, the need to motivate them is usually forgotten.

In this paper we give four situations that can serve to motivate complex numbers for students who have had two semesters of calculus. We have found that the best motivation for most new ideas is their utility in solving real problems. The examples presented here use complex numbers as a tool for obtaining real answers in real situations.

The mother of invention. Historically, complex numbers were introduced for practical reasons. Their use by Rafael Bombelli (1526–1572) provides insight into the need for complex numbers.

In the sixteenth century mathematicians were interested in finding solutions (real, of course) of polynomial equations. One of the high points [**3**] was Cardano's solution