Students find these methods a welcome change of pace from the routine numerical techniques. They enjoy the challenge to improve on bounds already obtained, and they gain valuable experience in working with inequalities, the heart of analysis.

Right Triangles with Perimeter and Area Equal

William Parsons, Borough of Manhattan Community College, New York, NY

Students learning about right triangles may observe that the area equals the perimeter for the 6, 8, 10 and for the 5, 12, 13 right triangles. The question naturally occurs whether this situation also holds for other right triangles whose legs have integral length. Thus, the following discussion may be of interest.

If a, b and $\sqrt{a^2 + b^2}$ are the sides of a right triangle, then the perimeter is $a + b + \sqrt{a^2 + b^2}$ and the area is $ab/2$. Equating both expressions yields

$$a^2 + b^2 = \frac{a^2b^2}{4} - ab^2 - a^2b + 2ab + a^2 + b^2,$$

which reduces to

$$a + b = \frac{ab}{4} + 2. \qquad (*)$$

Since a and b are natural numbers, it follows that $ab/4$ is a natural number and thus either 4 divides a or b, or 2 divides both a and b. In the latter case, letting $a = 2p$ and $b = 2q$, we find that $2p + 2q = pq + 2$. Therefore, 2 divides p or q, and so either a or b is divisible by 4. (Another elementary, but instructive, approach is the following: if neither a nor b is divisible by 4, then both a and b are even. Therefore, $(a/2)(b/2) = ab/4 = (a + b) - 2$ is even. But then either $a/2$ or $b/2$ is divisible by 2, and this contradicts the assumption that neither a nor b was divisible by 4.)

Suppose that $b = 4m$ for some positive integer m. Substituting in $(*)$, we get $a + 4m = am + 2$. Since this can be written as

$$(a - 4) + 2 = (a - 4)m,$$

we see that $(a - 4)$ divides 2. Thus, the only solutions to our problem occur for $a = 5$ and $a = 6$, with respective values $b = 12$ and $b = 8$.

Having come this far, instructors can introduce primitive Pythagorean triangles and raise the following conjecture:

For every natural number n, there is at least one primitive Pythagorean triangle in which the area equals n times the perimeter.

The case $n = 1$ yields the above cited 5, 12, 13 triangle. Now we may be motivated to try to verify this conjecture or to read the proof of Problem 3587 in *School Science and Mathematics* 76 (1976) 83–84.
