A computer algebra system or graphing calculator gives us approximate values for x and t leading to the points $(-0.608, 0.544)$ on the graph of f and $(-0.272, -0.074)$ on the graph of g. The distance between these points is 0.578.

To prove that our procedure is valid under the stated conditions, suppose we are given differentiable functions f and g for which the set $\{|PQ| : P = (x, f(x)), Q = (t, g(t))\}$ attains a minimum value $d = |P_0Q_0|$, where neither P_0 nor Q_0 is an endpoint of one of the graphs. We want to prove that the line segment P_0Q_0 is perpendicular to the tangent lines to f's graph at P_0 and to g's graph at Q_0. To do this, we hold the endpoint $Q_0 = (t_0, g(t_0))$ fixed and let $P = (x, f(x))$ vary. The square of the distance from Q_0 to P is the differentiable function $D(x) = (x - t_0)^2 + (f(x) - g(t_0))^2$ whose minimum value occurs where $D'(x) = 0$:

$$2(x - t_0) + 2(f(x) - g(t_0)f'(x) = 0.$$

So, $D(x)$ is minimum when $f'(x) = 0$ and $x = t_0$, or when $f'(x) \neq 0$ and

$$\frac{f(x) - g(t_0)}{x - t_0} = \frac{-1}{f'(x)}.$$

In either case, the segment P_0Q_0 of minimum length is perpendicular to f's graph at the point P_0. By reversing the roles of f and g, we get the other part of the claim and this completes the proof.

Of course, in practice, finding the lengths of line segments PQ that are perpendicular to the tangent lines at P and Q gives us only candidates for the distance between the graphs. We would need to determine whether the length of one them actually is the minimum distance between the two graphs [cf. c) below].

Suggested Problems

a) Find the minimum distance between the graphs of $f(x) = e^x$ and $g(t) = \ln t$.

b) Find the minimum distance between the graphs of $f(x) = 1 + (x + 1)^2$ and $g(t) = 1 - 1/t$ $(t > 0)$.

c) $f(x) = x + 2 + \sin x$ and $g(t) = t$.

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The Alternating Harmonic Series

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The following derivation that the alternating harmonic series converges to $\ln 2$ is more elementary than the standard one in the textbooks or the several that have appeared in journals (e.g., [**1**], [**2**], [**3**], [**4**], [**5**]) in that it does not use integrals or infinite series (except trivially).

Define

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n},$$

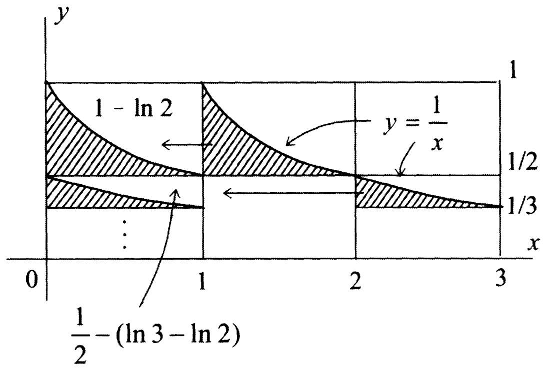

Figure 1.

and set

$$\gamma_n = H_n - \ln n, \quad \gamma_n' = H_{n-1} - \ln n.$$

The total area moved to the left is, after one step, $1 - \ln 2 = H_1 - \ln 2 = \gamma_2'$. After two steps it is $(1 - \ln 2) + (\frac{1}{2} - (\ln 3 - \ln 2)) = H_2 - \ln 3 = \gamma_3'$. And so on. The figure shows γ_n' as a sum of disjoint "triangular" areas on the interval [0, 1]. (This is our one use of integrals.) Since the sequence (γ_n') is increasing and bounded above (by 1), it converges to a limit. The sequence $(\gamma_n) = (\gamma_n' + \frac{1}{n})$ has the same limit. This limit is the famous *Euler's constant*, denoted by γ:

$$\gamma = \lim \gamma_n \lim(H_n - \ln n).$$

This number arises often in advanced mathematical analysis, and its value has been computed to a healthy number of decimal places. To three places, $\gamma = .577$. By the way, to this day no one knows whether γ is rational or irrational.

Let A (or A^2) denote the alternating harmonic series and A_n its nth partial sum. Since the terms of A alternate in sign and decrease in absolute value to zero, A converges to a limit; i.e., its sequence of partial sums (A_n) converges to that limit. Since every subsequence of (A_n), in particular the subsequence (A_{2n}), has the same limit as the parent sequence (A_n), it suffices to prove that $A_{2n} \to \ln 2$. (Similarly, $\gamma_{2n} \to \gamma$.) Writing out the terms shows that

$$A_{2n} = H_{2n} - H_n. \tag{1}$$

Then

$$\begin{aligned} A_{2n} &= [H_{2n} - \ln 2n] - [H_n - \ln n] + \ln 2n - \ln n \\ &\to \gamma - \gamma + \ln 2 = \ln 2. \end{aligned}$$

This derivation was outlined as a problem in the calculus text [**2**, p. 803], now out of print though probably available in some libraries.

For the generalization to $k > 2$, our point of departure will be the analogue of (1):

$$A_{kn}^k = H_{kn} - H_n. \tag{2}$$

For example,

$$A^3_{3n} = H_{3n} - H_n = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots + \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{2}{3n},$$

the $(3n)$th partial sum of the series

$$A^3 = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots.$$

To show that A^3 converges, we group the positive pairs and decompose the negative terms, as follows:

$$A^3 = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5}\right) - \cdots,$$

displaying A^3 as an alternating series whose terms decrease in absolute value to zero. The series A^3 therefore converges to a limit. Consequently, (A^3_{3n}), a subsequence of its sequence of partial sums, converges to the same limit. Arguing as in the original case, we find that

$$A^3_{3n} \to \ln 3 \text{ and, therefore, } A^3 \to \ln 3.$$

To see the pattern more fully, note that

$$A^4_{4n} = H_{4n} - H_n = 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \cdots.$$

Let A^k denote the series whose (kn)th partial sum is A^k_{kn}. The proof that $A^k \to \ln k$ goes the same way as the one just given for $k = 3$. (These extensions were noted in a successor manuscript to [**2**], which however died on the editor's desk.)

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Using Differential Equations to Describe Conic Sections

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"A *parabola* is the set of points in a plane that are equidistant from a fixed point F (called the *focus*) and a fixed line (called the *directrix*)." This definition is included in many calculus texts that have a chapter on analytic geometry. Using basic algebra