$$-15\binom{n-1}{2}2^{n-3}-31\binom{n-1}{3}2^{n-4}-53\binom{n-1}{4}2^{n-5}-82\binom{n-1}{5}2^{n-6}$$
$$-126\binom{n-1}{6}2^{n-7}-168\binom{n-1}{7}2^{n-8}\bigg\}.$$

This formula is established for $n \geq 11$ by an inclusion–exclusion argument (details available from the author), together with the facts that

$$\binom{n-1}{1}\binom{n-2}{k}=\binom{k+1}{1}\binom{n-1}{k+1} \quad \text{and} \quad \binom{n-1}{5}\binom{n-6}{k}=\binom{k+5}{5}\binom{n-1}{k+5}$$

and then checked to be true for $1 \leq n \leq 10$ also.

If we multiply by $n$, use the fact that

$$n\binom{n-1}{k}=(k+1)\binom{n}{k+1}$$

and sum over $n \geq 0$, we find that

$$E(X)=\sum_{n\geq 0} np_n=\frac{441357301}{11943936},\ L=12+E(X)=\frac{584684533}{11943936}. \quad \blacksquare$$

**Reference**


1. Min Deng and Mary T. Whalen, The mathematics of *Cootie*, this JOURNAL, 29 (1998), 222–224.


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## An "Extremely" Cautionary Tale


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This is a tale of something that might be about to happen to you, as it did to me not too long ago. If so, you are teaching multivariable calculus, as you have often done before, and it is time to make up an exam. This particular exam should cover topics such as directional derivatives, gradient, maxima and minima, and Lagrange multipliers. Since only one class period is available, the computations had better be fairly straightforward. In particular, the Lagrange multiplier problem should probably be similar to one that the students are (supposed to be) familiar with.

One of the assigned homework problems, number 10 on p. 730 of [**1**], reads as follows: "Find the least distance between the origin and the surface $x^2y - z^2 + 9 = 0$." This is a fine problem, but the numbers are a bit unwieldy; the $x$-coordinates of the closest points (on the surface) to the origin turn out to be $\pm\sqrt[6]{162}$. So you decide to replace 9 by $k$ and then see how to choose $k$ to streamline the computation. While you are at it, you interchange $x$ and $y$ to make the problem look slightly different.

Of course, *you* have no trouble minimizing $f(x, y, z) = x^2 + y^2 + z^2$ under the constraint $g(x, y, z) = xy^2 - z^2 + k = 0$, using Lagrange multipliers. Here is a sketch of your computation (soon you will be checking this very carefully, in view of later events, but all the algebra is indeed correct):

$$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow 2x = \lambda \cdot y^2,\ 2y = \lambda \cdot 2xy,\ 2z = \lambda \cdot (-2z)$$
$$\Rightarrow \lambda = -1 \text{ (Case 1) or } z = 0 \text{ (Case 2)}.$$

In Case 1 we have $2x = -y^2$, $2y = -2xy$, $xy^2 - z^2 + k = 0$. The equation $2y = -2xy$ leads to the subcases $y = 0$ and $x = -1$, and we end up with the six possible points $(0, 0, \pm\sqrt{k})$ and $(-1, \pm\sqrt{2}, \pm\sqrt{k-2})$. In Case 2 we have $2x = \lambda \cdot y^2$, $2y = \lambda \cdot 2xy$, $xy^2 + k = 0$. If $y = 0$ we get $x = 0$, $k = 0$, and so the point $(0, 0, 0)$ we find is not new; if $y \neq 0$ we get $\lambda = \frac{1}{x}$, which leads to the two new points $(-\sqrt[3]{k/2}, \pm\sqrt{2}\sqrt[3]{k/2}, 0)$.

By now you see that to get a nice answer, $k = 2$ is a good choice. For $k = 2$ the points found become $(0, 0, \pm\sqrt{2})$ and $(-1, \pm\sqrt{2}, 0)$; the distances from these points to the origin are $\sqrt{2}, \sqrt{3}$ respectively, so $\sqrt{2}$ is the least distance.

To check your answer, you decide to redo the problem without Lagrange multipliers. After all, if $xy^2 - z^2 + 2 = 0$, then $z^2 = xy^2 + 2$, so $x^2 + y^2 + z^2 = x^2 + y^2 + xy^2 + 2$, and so we should find the points closest to the origin if we minimize $h(x, y) = x^2 + y^2 + xy^2 + 2$. A routine computation starting with $\partial h/\partial x = \partial h/\partial y = 0$ yields $(0, 0)$ and $(-1, \pm\sqrt{2})$ as the critical points of the function $h$, and the answer above is confirmed.

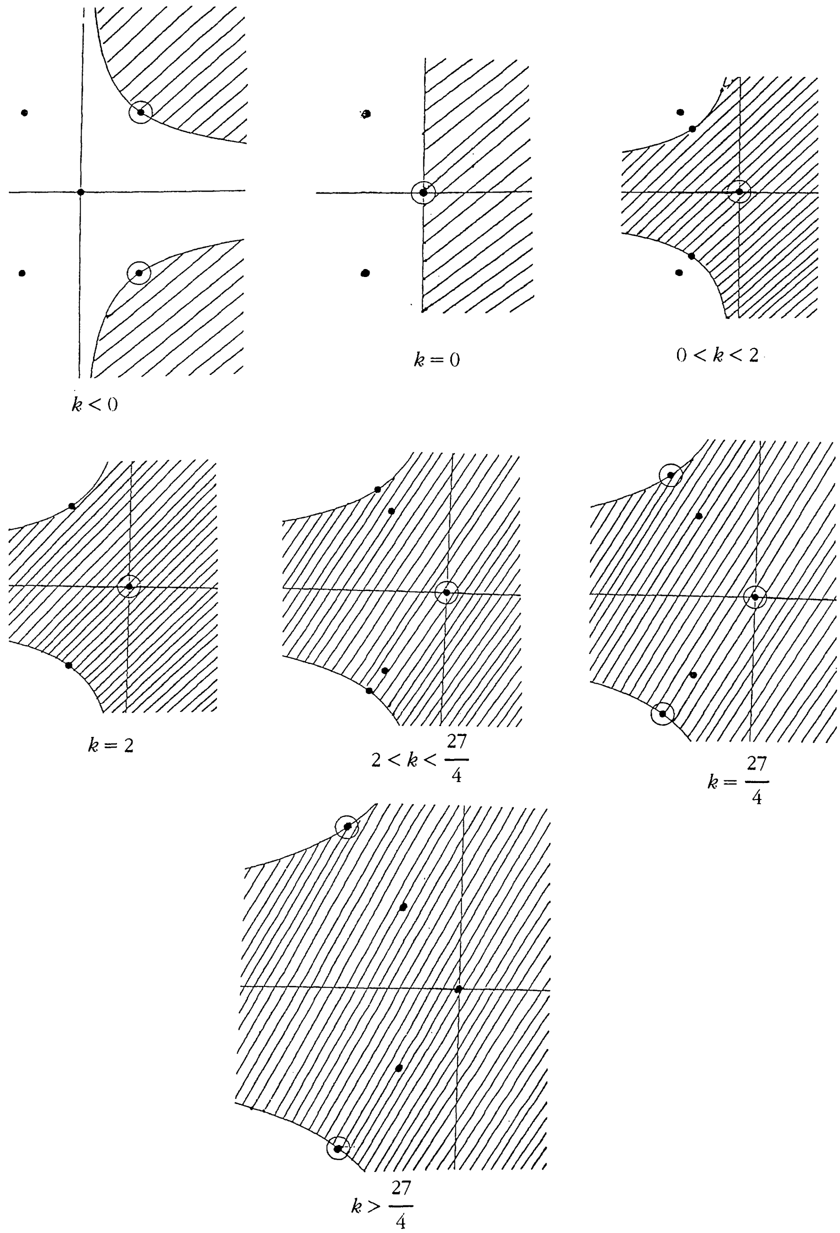
All right, confession time. This isn't quite what happened to me—and one of the two methods above is *wrong*, as we'll see shortly. What actually happened is that I chose $k = 16$ instead of $k = 2$. The "candidate points" found using the Lagrange multiplier methods were, therefore, $(0, 0, 4)$, $(-1, \pm\sqrt{2}, \pm\sqrt{14})$, and $(-2, \pm 2\sqrt{2}, 0)$. Since the distances from these points to the origin are 4, $\sqrt{17}$, $\sqrt{12} = 2\sqrt{3}$ respectively, this computation gave $2\sqrt{3}$ as the least distance from the origin to $xy^2 - z^2 + 16 = 0$.

On the other hand, the check without Lagrange multipliers, relying on the substitution $z^2 = xy^2 + 16$, yielded essentially the same function to minimize as for $k = 2$, specifically $h(x, y) = x^2 + y^2 + xy^2 + 16$; only the constant term had changed. So $h$ still had the critical points $(0, 0)$ and $(-1, \pm\sqrt{2})$, and since $h(0, 0) < h(-1, \pm\sqrt{2})$, this method gave $\sqrt{h(0, 0)} = 4$ as the least distance from the origin to $xy^2 - z^2 + 16 = 0$.

Obviously, something was wrong somewhere, and this was the point at which I spent at least twenty minutes checking my algebra, to no avail.

As I realized eventually, the mistake is in the second method. The trouble is that when we substitute $z^2 = xy^2 + k$ into $x^2 + y^2 + z^2$ and then look for the minimum of $h(x, y) = x^2 + y^2 + xy^2 + k$, we lose information. Specifically, we lose track of the fact that $xy^2 + k$, being equal to $z^2$, must be nonnegative. So we should really look for the minimum of $h(x, y)$ *on the region* $R = \{(x, y) | xy^2 + k \geq 0\}$. Now it is easy to see that, depending on the value of $k$, this minimum may occur either at a critical point of $h(x, y)$ or at a point on the boundary curve $xy^2 + k = 0$. The figure illustrates what actually happens for various values of $k$. In each case, the region $R$ is shaded, the critical points $(0, 0)$ and $(-1, \pm\sqrt{2})$ of $h(x, y)$ are shown whether or not they are in $R$, along with the boundary points $(-\sqrt[3]{k/2}, \pm\sqrt{2}\sqrt[3]{k/2})$, and those point(s) where $h(x, y)$ has its minimum on $R$ are circled.

The moral of this cautionary tale: If you are looking for an extremum of $f(x, y, z)$ under a constraint $g(x, y, z) = 0$ and you do so by using the constraint to rewrite




$f(x, y, z)$ as $h(x, y)$, which reduces the problem to a two-variable one, make sure to look for the extremum of $h(x, y)$ in the region of the $x, y$-plane consisting of only those points $(x, y)$ for which there actually exists a number $z$ with $g(x, y, z) = 0$.

## Reference

1. D. Varberg and E. J. Purcell, *Calculus with Analytic Geometry*, 6th ed., Prentice Hall, 1992.