[5] Martin Gardner, More about tiling the plane: the possibilities of polyominoes, polyiamonds, and polyhexes, Sci. Amer. (August 1975) 112–115.
[6] F. Göbel, Geometric Packing and Covering Problems, Packing and Covering In Combinatorics, A. Schrijver (Editor), Mathematical Centre, Amsterdam, 1979, pp. 179–199.
[7] Branko Grünbaum and G. C. Shephard, The eighty-one types of isohedral tilings in the plane, Math. Proc. Cambridge Philos. Soc., 82 (1977) 177–196.
[8] H. Heesch and O. Kienzle, Flächenschluss, Springer-Verlag, Berlin, 1963.
[9] P. A. MacMahon, New Mathematical Pastimes, Cambridge Univ. Press, 1930, p. 80ff. (To be reprinted in Percy Alexander MacMahon: Collected Papers, vol. 2, George E. Andrews (Editor), MIT Press.)
[10] Ivan Niven, Convex polygons that cannot tile the plane, Amer. Math. Monthly (December 1978) 785–792.
[11] E. R. Ranucci and J. L. Teeters, Creating Escher-Type Drawings, Creative Publications, Palo Alto, California, 1977, p. 83ff.
[12] Doris Schattschneider, The plane symmetry groups: their recognition and notation, Amer. Math. Monthly (June-July 1978) 439–450.

Nice Cubic Polynomials for Curve Sketching

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The polynomials used in calculus texts to illustrate the applications of derivatives to curve sketching are, in spite of any student impressions to the contrary, reasonably nice polynomials. Most of their coefficients are integers, relatively small integers at that, and it is not unusual for the polynomials or even the derivatives to have an integer root. This is no mere coincidence. To simplify the sketching of a given polynomial, students are limited to such devices as a careful choice of origin, scale or coordinate system. Authors can use the additional device of careful selection of the polynomial. According to their taste, roots can be integers, rationals or even (cruelty!) irrationals. What if integer roots are chosen? When it comes time to find the point of inflection and relative extremes of cubic polynomials, the odds favor relative extremes at quadratic irrationals and points of inflection at rationals that are not integers.

It is the purpose of this note to use Pythagorean triplets to show how to construct cubic polynomials with integer coefficients whose roots are integers and whose first and second derivatives also have integer roots. We do not claim that this construction is original, but it is not such common knowledge that repeating it here will not be useful when it comes time to make up a calculus test.

We can suppose that the roots of the polynomial are 0, a, and b, since if one of the roots is not 0, a translation can make it so. Let

$$f(x)=x(x-a)(x-b)=x^3-(a+b)x^2+abx,$$

so

$$f'(x)=3x^2-2(a+b)x+ab \quad \text{and} \quad f''(x)=6x-2(a+b). \tag{1}$$

If

$$f'(x)=3(x-c)(x-d)=3x^2-3(c+d)x+3cd, \tag{2}$$

then comparing (1) and (2) gives

$$2(a+b)=3(c+d) \quad \text{and} \quad ab=3cd. \tag{3}$$

From the second equation, one of a and b is divisible by 3 and because of symmetry, we can suppose that it is a. From the first equation, $a+b$ is divisible by 3, and so b is divisible by 3 also. Putting $a=3A$ and $b=3B$, (3) becomes

$$2(A+B)=c+d \quad \text{and} \quad 3AB=cd.$$

From the second equation, one of c and d is divisible by 3, and we can suppose that it is c. If $c=3C$, we have

$$2(A+B)=3C+d \quad \text{and} \quad AB=Cd. \tag{4}$$

Eliminating A from (4) yields $2(Cd/B+B)=3C+d$, or

$$2B^2-(3C+d)B+2Cd=0. \tag{5}$$

If B is to be an integer, the discriminant of the quadratic in (5) must be a perfect square:

$$(3C+d)^2-16Cd=H^2$$

which can be rearranged to

$$H^2+(4C)^2=(d-5C)^2.$$

We know how to find all of the Pythagorean triplets satisfying the above equation:

$$4C=2kst, \quad H=k(s^2-t^2), \quad d-5C=k(s^2+t^2) \tag{6}$$

for integers s, t and any k. From (5), $B=(3C+d\pm H)/4$. Taking the plus sign (it turns out that choosing the minus sign gives the same result with s and t interchanged), (6) gives

$$\begin{aligned} B&=(3C+5C+k(s^2+t^2)+k(s^2-t^2))/4 \\ &=(4kst+2ks^2)/4. \end{aligned}$$

Putting $k=2r$ we get $B=rs(s+2t)$. From (6), $C=rst$ and $d=5rst+2r(s^2+t^2)=r(2s+t)(s+2t)$. Finally, substituting back we get

$$\begin{aligned} a&=3rt(2s+t) \\ b&=3rs(s+2t) \\ c&=3rst \\ d&=r(2s+t)(s+2t). \end{aligned}$$

The root of the second derivative is

$$(a+b)/3=r(s^2+4st+t^2).$$

TABLE 1 gives some examples of these nice cubics; the list can be indefinitely extended.

Polynomial	Roots of		
f	f	f'	f''
x^3-3x^2+4	$-1,2,2$	$0,2$	1
x^3-33x^2+216x	$0,9,24$	$4,18$	11
x^3-6x^2-135x	$-9,0,15$	$-5,9$	2
$x^3-147x+286$	$-13,2,11$	$-7,7$	0
$x^3-3x^2-144x-140$	$-10,-1,14$	$-6,8$	1
$x^3-3x^2-144x+432$	$-12,3,12$	$-6,8$	1

TABLE 1

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