Young Students Explore Proofs Without Words

By James Tanton, with the help of Chiron Anderson, Shivani Angappan, Adam Cimpeanu, Kevin Dibble, Theo Fitzgerald, Bianca Homberg, Steven Homberg, Eric Marriott, Curtis Mogren, Alexandra Palocz, Linus Schultz, William Sherman, Alex Smith, David Tang, Steven Tang, Andrew Ward

Looking for a discussion topic for your math club? A tidbit to fill an idle moment in class? Try presenting a Proof Without Words — or two or three — as published in the journals of the MAA. You might be surprised by the conversations that follow!

In the spring of 2009 I presented young students (ages 9–17) of the St. Mark's Institute of Mathematics some wordless demonstrations of the Galilean ratios:

$$\frac{1}{3}=\frac{1+3}{5+7}=\frac{1+3+5}{7+9+11}=\frac{1+3+5+7}{9+11+13+15}=\cdots$$

Within minutes these youngsters came up with their own Proof Without Words of the result:

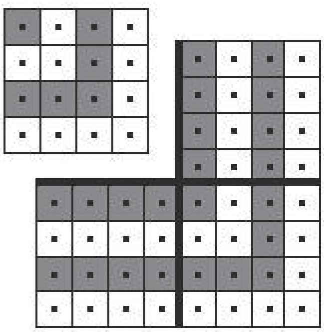

Figure 1

$$\frac{\text{The sum of the first k odds}}{\text{The sum of the next } k \text{ odds}}=\frac{1}{3}$$

And that was only the beginning!

Generalization 1: Students noted that a square also divides into 9 parts, 16 parts, and so on, and we can say more generally:

$$\frac{\text{The sum of the first } k \text{ odds}}{\text{The sum of the next } mk \text{ odds}}=\frac{1}{(m+1)^2-1}$$

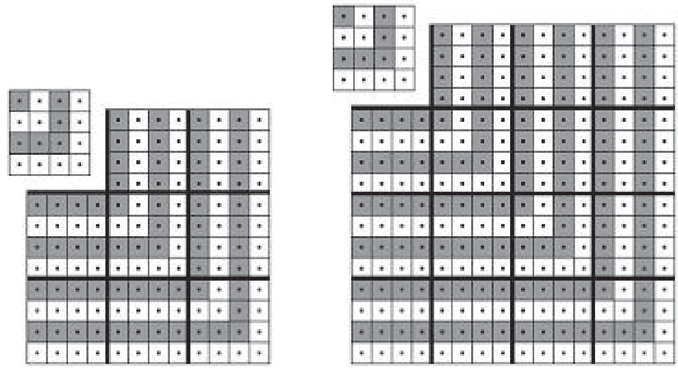

Figure 2

For example,

$$\frac{1}{3+5}=\frac{1+3}{(5+7)+(9+11)}=\frac{1+3+5}{(7+9+11)+(13+15+17)}\cdots=\frac{1}{8}$$

and

$$\frac{1}{3+5+7}=\frac{1+3}{(5+7)+(9+11)+(13+15)}\cdots=\frac{1}{15}$$

Generalization 2: The choice of coloring was fortuitous and inspired more. If we adjust the shading of squares of odd dimension and regard the gray cells as positive and white cells as negative, we discover an alternating version of the Galilean ratios.

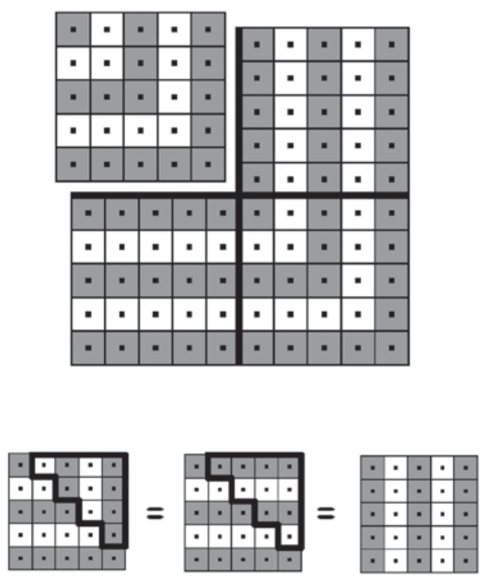

Figure 3

$$\frac{1}{3}=\frac{1-3+5}{7-9+11}=\frac{1-3+5-7+9}{11-13+15-17+19}=\cdots$$

Alternatively, if we regard the grey squares as 'worth' a and the white squares as b we obtain:

$$\frac{a}{3b}=\frac{a+3b+5a}{7a+9b+11a}=\frac{a+3b+5a+7b+9a}{11a+13b+15a+17b+19a}=\cdots$$

[What results follow from using more than two colors?]

Bonus: Our side picture for odd squares establishes

$$\frac{3}{1+5}=\frac{1}{2},\ \frac{3+7}{1+5+9}=\frac{2}{3},\ \frac{3+7+11}{1+5+9+13}=\frac{3}{4}$$

and, in general $\frac{3+7+\cdots+(4n-1)}{1+5+\cdots+(4n+1)}=\frac{n}{n+1}$.

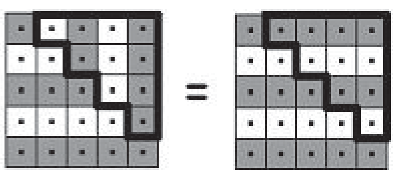

Figure 4

Generalization 3: A path for more results is clear. Pictures of the type:

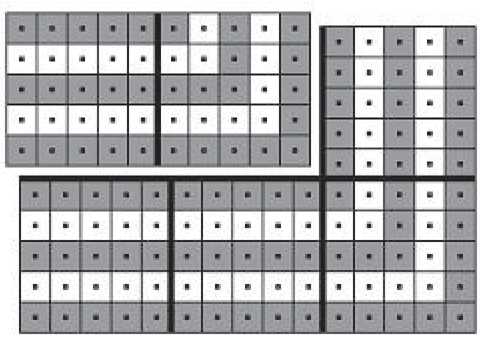

Figure 5

show, for instance,

$$\frac{2}{4}=\frac{3+5}{7+9}=\frac{4+6+8}{10+12+14}=\frac{5+7+9+11}{13+15+17+19}=\frac{6+8+10+12+14}{16+18+20+22+24}=\cdots$$

And increasing the number of rows and columns in this picture (akin to generalization 1) yield a whole host of similar results.

If we reduce the fractions with even terms we obtain:

$$\frac{1}{2}=\frac{2+3+4}{5+6+7}=\frac{3+4+5+6+7}{8+9+10+11+12}=\cdots$$

which possesses its own Proofs Without Words:

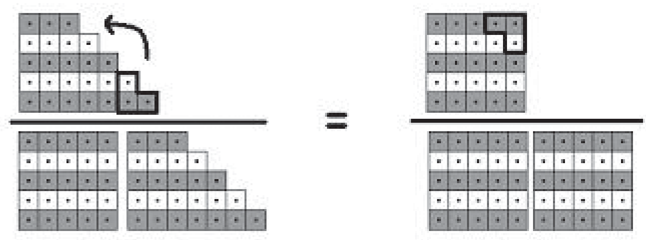

Figure 6

And we also see an alternating version of the result if we again regard grey and white cells as opposite in parity. [Add rows of blocks to this diagram. What more can be discovered?]

Concluding Comments: This story of pictures took place over just a few 40 minute sessions and demonstrates the depth that can be readily attained from intellectual play. And from my experience students of all ages are keen to engage in play when given the chance! Proofs Without Words provide a comfortable and appealing gateway to original enquiry.

Jim Tanton and his collaborators are at the St. Marks Institute of Mathematics; see http://www.stmarksschool.org/academics/mathinstitute.aspx.

Call for Proposals

All MAA members are invited to consider submitting proposals for contributed paper sessions, minicourses, and panels at the two annual meetings of the MAA. Information about procedures and deadlines may be found on the MAA website at http://www.maa.org/meetings/meetings.html. The first MathFest deadlines are in October (10 months before the meeting) and the first Joint Mathematics Meetings deadlines are in December (13 months before the meeting).