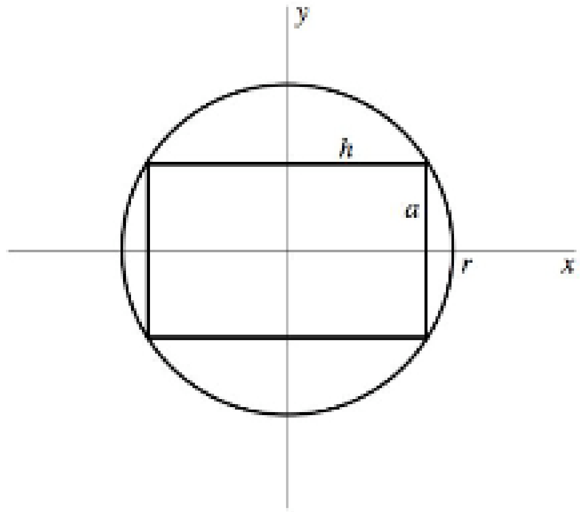
Volume of a napkin ring



The diagram shows a central cross-section of a sphere of radius r through which a centrally placed circular cylinder of radius a has been drilled and the material removed. The shape left is, for obvious reasons, often called a "napkin ring."

To compute the volume, V, of material in the napkin ring, we first observe that

$$V = \text{vol of sphere} - \text{vol of cylinder} - 2 \times \text{vol of one of the spherical caps}$$

The following formulas are standard:

$$\text{vol of sphere} = \tfrac{4}{3}\pi r^3$$

$$\text{vol of cylinder} = 2\pi h a^2$$

By calculus,

$$\text{vol of a cap} = \int_h^r \pi(r^2 - x^2)\,dx = \pi\left[r^2 x - \frac{x^3}{3}\right]_h^r$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2 h - \frac{h^3}{3}\right)\right] = \pi\left[\frac{2r^3}{3} - r^2 h + \frac{h^3}{3}\right]$$

Hence,

$$\text{vol of ring} = \pi\left[\tfrac{4}{3}r^3 - 2ha^2 - \tfrac{4}{3}r^3 + 2r^2 h - \tfrac{2}{3}h^3\right]$$

$$= \pi\left[2r^2 h - 2ha^2 - \tfrac{2}{3}h^3\right] = \pi\left[2r^2 h - 2h(r^2 - h^2) - \tfrac{2}{3}h^3\right]$$

$$= \pi\left[2r^2 h - 2hr^2 + 2h^3 - \tfrac{2}{3}h^3\right] = \tfrac{4}{3}h^3$$

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