Pursuit Curves for the Man in the Moone

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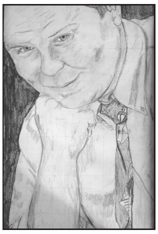

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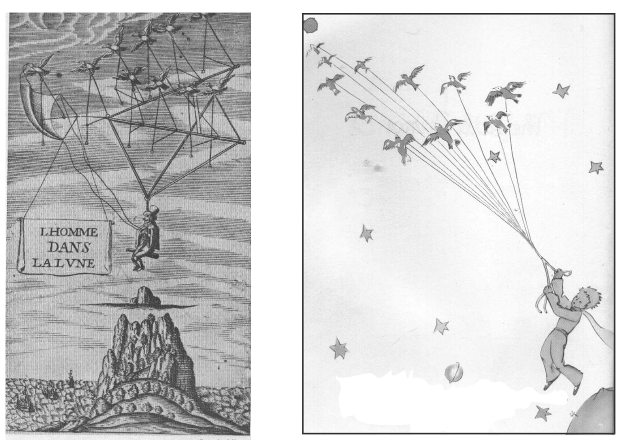
In 1638, Francis Godwin's story *The Man in the Moone* was published. In that story an astronaut harnesses a wedge of 25 swans, as suggested in Figure 1(a) [**3**, plate facing p. 118], and flies to the moon. The swans fly at a constant rate and always head toward the moon, in accordance with their annual migration. Thus, the trajectory from the earth to the moon is not a straight line, but is a pursuit curve as shown in Figure 2. In the story, Godwin says that the flight to the moon takes twelve days, whereas the return trip, which follows a straight line, takes eight days. How good a guess was Godwin's?



(a) via a wedge of swans (b) via a flock of wild birds

Figure 1. Space travel.

We use a little calculus to determine the consistency of these flight times. We also explore some popular fancies regarding motion from the literature of yesteryear.

A little history

Let's put Godwin's guess into historical context, and compare it with some other guesses. In each of the following narratives, unless otherwise stated, the tacit assumption is that the return trip is as long as the outbound trip.

Lucian, in the comic satire *The True History*, written in 174 AD, describes a ship whirled aloft by a waterspout, whereafter it sails eight days to the moon. In 1300, Dante travels through the solar system in *The Divine Comedy*. Together with his guide Beatrice, they leave earth from a mountaintop, and fall upwards. To describe their speed, Beatrice says to Dante,

> For lightning ne'er ran, as thou hast hither, [*Paradisio*, canto i].

In a matter of minutes, they arrive on the moon. Around 1610, Johannes Kepler wrote *The Dream*, in which a hero is transported by a force field to the moon in four hours. Around 1630, Galileo calculated that a ball would fall from the moon to the earth in 3 hours, 22 minutes, and 4 seconds [**5**, p. 224]. In 1640, John Wilkins, the first secretary of the Royal Society, anticipated that a manned moon trip would last 180 days. In 1656, in *Voyage to the Moon*, Cyrano de Bergerac, famous for his long nose and his swordsmanship, initially launching a fictional version of himself in a winged contraption powered by fireworks, travels there and back, each leg lasting several days. In 1835, in *The Adventures of Hans Pfaall*, Edgar Allan Poe describes a two-week-long moon voyage in a balloon inflated with gas "37.4 times lighter than hydrogen." In 1870, in *Round the Moon*, Jules Verne describes a round-trip projectile shot to the moon, each leg lasting four days. In 1901, in *The First Men in the Moon*, H. G. Wells describes an anti-gravity moon flight of a few days with a return trip lasting several weeks. In 1968, Apollo 8, the first manned craft to circle the moon, reached the moon in 3.5 days, and returned in 2.5 days.

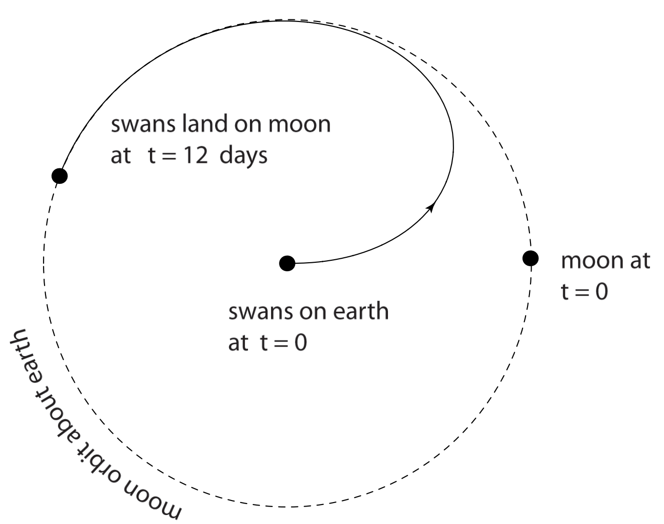


Figure 2. The trajectory from the earth to the moon.

Godwin (1562–1633), an Anglican bishop, is primarily known as a church historian. He wrote his moon story around 1599, and circulated it privately among friends—the story builds on Copernican theory, and publication would have caused him needless controversy. He chose a Spanish background for his astronaut, Domingo Gansales, so as to to be in the spirit of the Spanish discovery of America under Columbus a hundred years earlier. After Godwin died, a friend published the story anonymously. It went through many printings in the next two centuries and inspired similar stories. For example, in *The Little Prince*, a classic children's story, the Little Prince leaves his planet for earth by harnessing a flock of birds, as shown in Figure 1(b) [**4**, p. 2].

With respect to the swans flying through the heavens, Domingo and the swans start their journey from a mountaintop, just as in Dante's story. For the first few moonward miles, the birds strain at transporting Domingo. But thereafter, earth's pull of gravity vanishes. The birds have an easy time until just before landing on the moon, when gravity again becomes an observable phenomenon.

Domingo's trajectory to the moon is a pursuit problem, a puzzle with a long tradition going back to Zeno's mythic race of Achilles overtaking the tortoise. The earliest Latin version of the problem dates to the court of Charlemagne, where the scholar Alcuin collected some mathematical conundrums to sharpen young minds. The 26th of these reads,

> There is a field which is 150 feet long. At one end stood a dog, at the other a hare. The dog advanced to chase the hare. Whereas the dog went nine feet per stride, the hare went seven. How many feet and how many leaps did the dog take in pursuing the fleeing hare until it was caught? [**10**]

Figure 3. Reynard the Fox in pursuit of the rabbit, drawn by J. J. Mora (1901).

Later versions of this recreational math problem vary the nature of the pursued and the pursuer. Figure 3 shows a fox chasing a hare, from the epic satirical poem *Reynard the Fox* of 1483 [**1**, plate facing p. 94]. A popular renaissance version includes couriers overtaking one another on roads between various Italian cities [**11**, p. 70]. The first to apply Newton's calculus to pursuit problems was Pierre Bouguer in 1732, who imagined a privateer overtaking a merchant ship on the high seas [**2**, p. 91]. More recent versions have a fighter plane overtaking a bomber and a missile overtaking a jet and the space shuttle overtaking the Hubble Telescope. Solutions to Bouguer's problem appear in many differential equations texts such as Simmons [**9**, pp. 42–44]. In particular, a rabbit, initially at the origin, runs along the y-axis at constant speed a while pursued by a fox, initially out along the x-axis, who always runs towards the rabbit at constant speed b. The fox's path towards the rabbit is called a tractrix, and, if $b > a$, the fox eventually overtakes the rabbit. Much information about pursuit curves can be found on the web (for example, `http://mathworld.wolfram.`

com and the National Curve Bank site at http://curvebank.calstatela.edu/pursuit/pursuit.htm).

Pierre-Louis Moreau de Maupertuis is credited with generalizing the pursuit problem in 1735, so that the pursued can follow paths other than straight lines. In fact, the pursuit problem of Maupertuis commonly refers to the pursuit problem when the pursued travels along a circular arc. Versions of his problem have appeared several times in early volumes of the *American Mathematical Monthly*, including this:

> A dog at the center of a circular pond makes straight for a duck which is swimming along the edge of the pond. If the rate of swimming of the dog is to the rate of swimming of the duck as n is to 1, determine the equation of the curve of pursuit and the distance the dog swims to catch the duck.

Summarizing their limited success at solving this problem in the first volume of the *Monthly* in 1894, one of four solvers concluded that the problem solution "transcends the present limits of mathematical genius" [**2**]. In 1921, Hathaway [**6**] and Morley [**8**] gave detailed approximate solutions to the duck-and-dog problem. We use a similar approach in presenting a solution to Godwin's problem, and add a new twist in that, contrary to the terrestrial problem, the celestial one involves both the pursued and the pursuer rotating about a common center in good Copernican fashion.

An earth-moon model of the journey

To model Domingo's flight, position the earth at the origin, O. Assume that the moon orbits the earth counterclockwise in a circle with period $p = 27.3$ days and radius $D = 1$ lunar unit (LU), where 1 LU $\approx$ 384,000 km. We take p as the sidereal period (the period with respect to the fixed background of stars) of 27.3 days rather than the synodic period (the period with respect to the sun) of 29.5 days. Figure 4 shows the geometry of these two terms. (In 29.5 days, the moon goes around a circle of one lunar unit about 1.08 times.)

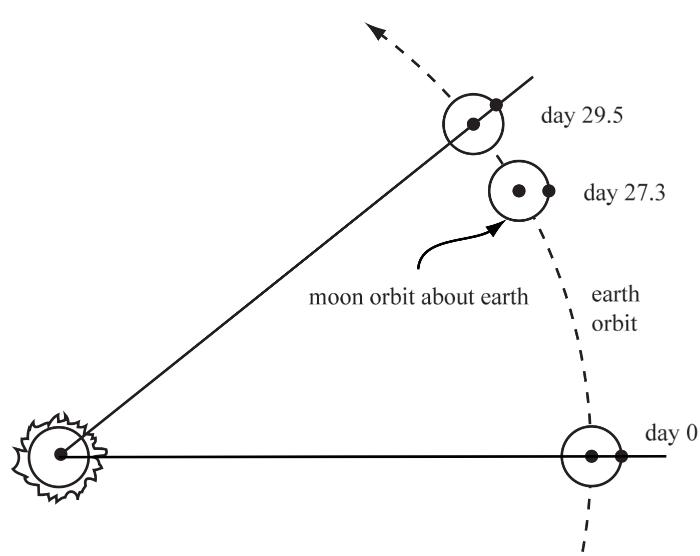


Figure 4. Sidereal versus synodic moon periods.

Assume that at time $t = 0$, the moon is at $(1, 0)$. The center of the moon at any time is given by

$$M(t) = (M_1(t), M_2(t)) = \left(\cos \frac{2\pi t}{p}, \sin \frac{2\pi t}{p}\right). \tag{1}$$

In terms of lunar units, earth's radius is $R \approx 0.0167$, and the moon's radius is $r \approx 0.00453$ (since the earth and moon radii are about 6400 km and 1738 km, respectively). Let $x(t)$ and $y(t)$ denote Domingo's position along his trajectory at time t. Assume that at $t = 0$, Domingo and his wedge of swans are on the earth's surface at $(R, 0)$. Since the swans always fly in the direction of the moon, these directions are also the directions of the trajectory's tangent line at each moment. Routine computation gives the differential equation system

$$\begin{cases} (M_2 - y)\, x' = (M_1 - x)\, y', \\ (x')^2 + (y')^2 = c^2. \end{cases} \tag{2}$$

Thus, one way to generate Domingo's pursuit curve to the moon is to solve (2), along with the initial condition $(x(0), y(0)) = (R, 0)$. This system is valid until the swans reach the surface of the moon.

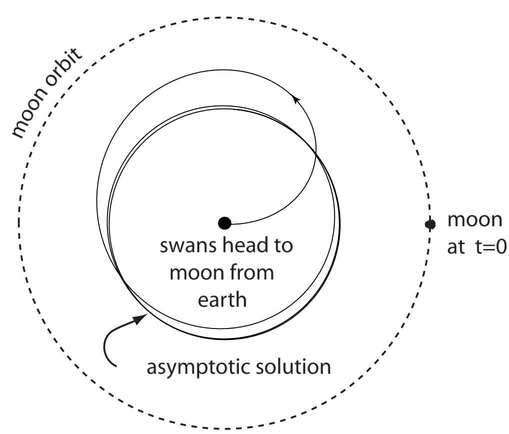


Figure 5. A hopeless quest, $c = 0.15$ lunar units/day.

As long as the swans' cruising speed c exceeds the speed k of the moon, the swans will ultimately reach the moon. However, if $c \leq k$, their trajectory will asymptotically approach a circle with center O and radius $\rho = cp/(2\pi)$. For example, as shown in Figure 5 with $c = 0.15$, the swans initially get about three-fourths of the way to moon orbit before falling back into a circular orbit about half-way to the moon. In particular, the asymptotic solution to (2) for any initial condition with $c < k$ (except of course when the swans are already on the moon) is

$$(x(t), y(t)) = \rho\left(\cos\left(\frac{2\pi t}{p} + \theta\right), \sin\left(\frac{2\pi t}{p} + \theta\right)\right), \tag{3}$$

where θ is the phase angle $\cos^{-1}(\rho/D)$. As can be verified, (3) satisfies (2).

To see why (3) is a solution, see Figure 6. The swans at s see the moon at m. Since the periods of the moon and the swans are the same, the tangent lines to the inner circle

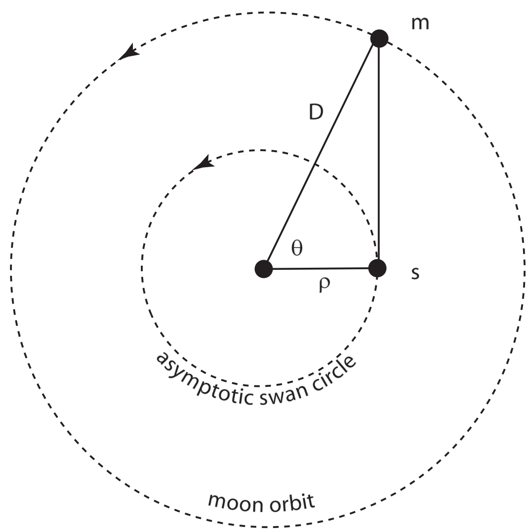


Figure 6. Finding the phase angle.

at the swans' position always go through the moon. Thus the phase angle between (1) and (3) is θ for all t.

Because standard numerical routines in a typical CAS may have difficulty when x' or y' is zero and because (2) ambiguously describes the motion of the swans as going away from the moon as well as towards it, we use a two-variable form of Euler's method that generates valid trajectories. The graphs of Figures 2 and 5 were generated in this way. Allow the swans to alter their direction every Δt time units. Let $t_n = n\,\Delta t$, where n is a positive integer. Suppose that Domingo is at position (x_n, y_n) at time t_n. The direction of the swans' flight is thus

$$(X, Y) = (M_1(t_n) - x_n,\ M_2(t_n) - y_n).$$

Let $\delta = \sqrt{X^2 + Y^2}$. Then $(X, Y)/\delta$ is the unit vector in the direction in which the swans fly. Their velocity is therefore $(X, Y)c/\delta$. The changes in the coordinates of Domingo's position from t_n to t_{n+1} are $\Delta x = c\,X\,\Delta t/\delta$ and $\Delta y = c\,Y\,\Delta t/\delta$. Let

$$(x_{n+1}, y_{n+1}) = (x_n + \Delta x, y_n + \Delta y). \tag{4}$$

Iterate (4) until δ is less than r, the radius of the moon. Then draw a path through the points (x_n, y_n).

Here is a *Mathematica* procedure that implements these ideas.

```
swan[Δt_, c_]:= Block[{x, y, X, Y, δ, t, bag, flag},
    x=R; y=0; bag={{x, y}}; flag=True; t=0;
    While[flag, X=M1[t]−x; Y=M2[t]−y;
        δ = √(X² + Y²); If[(δ <r) || (t>100), flag=False];
        x=x+X c Δt/δ; y=y+Y c Δt/δ; t=t+Δt; bag=Append[bag, {x, y}]];
    Print["there: ",t," days, back: ",(1-R-r)/c," days"];
    ListPlot[bag, PlotJoined→True, AspectRatio→Automatic]];
```

When using swan, the step-size Δt should be chosen so that $c\,\Delta t < r$ to ensure that the swans do not overshoot the moon. Furthermore, procedure swan ceases when it generates a point within r units of the moon's center (since the trip is successful once the swans reach the moon's surface) or if time t exceeds 100 days (a cut-off time chosen for convenience). The penultimate line in swan generates a text message giving the times to the moon and back again.

With R, r, p, M_1[t], and M_2[t] already defined, using swan[0.01, 0.2508] generates Figure 2 with a flight time of 12.01 days, (for a speed of $c = 0.2508$ LU/day), and a return flight time of 3.90 days, rather than Godwin's estimate of 8 days. In general, the duration of the return flight is $(1 - R - r)/c$. Furthermore, as c increases from k, the ratios of the time lengths of there and back again decrease from ∞ to 1, as illustrated in Figure 7. This graph should make intuitive sense, because as the swans' speed increases, their trajectory to the moon approaches a straight line, whereas if their speed is close to k, their trajectory spirals in multiple loops that converge towards the circle of the moon's orbit.

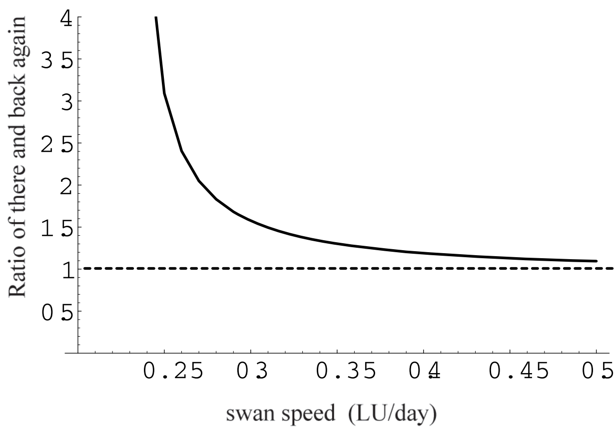


Figure 7. The ratio of travel times to the moon and back again.

A sun-earth-moon model of the journey

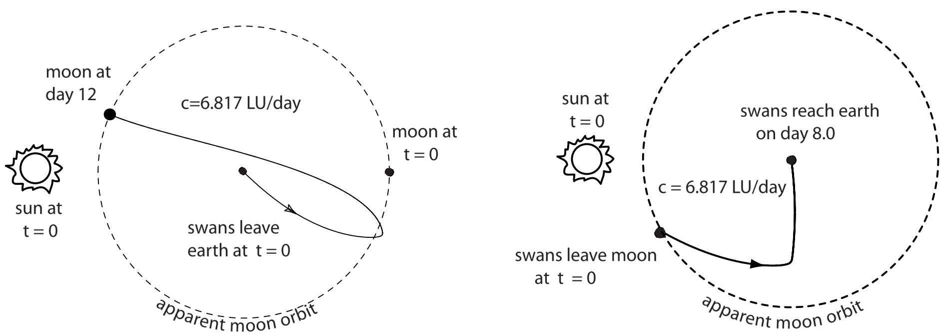
Since Godwin describes himself as a Copernican—"I joyne in opinion with Copernicus" [**7**, p. 21]—let's see if adding the sun into the model affects the journey lengths to the moon and back. This time, let the sun be at the origin. Assume that the earth moves around the sun in a circle with period $q = 365$ days and radius $Q = 390.625$ LU, since the earth is about 150,000,000 km from the sun. Then the center of the earth is at

$$G(t) = (G_1(t), G_2(t)) = Q\left(\cos\frac{2\pi t}{q}, \sin\frac{2\pi t}{q}\right).$$

In order to explore more fully the possible range of trajectories to the moon and back in the sun-earth-moon model, we let the center of the moon be at

$$\mu(t, T) = (\mu_1(t, T), \mu_2(t, T)) = G(t) + M(t + T), \tag{5}$$

where $0 \leq T \leq p$. In this model, the speed of the earth with respect to the sun is $K = 2\pi Q/q \approx 6.72$ LU/day, while the speed of the moon with respect to the sun is not constant, fluctuating slightly above and below K. As long as the swans fly faster than $K + k$ with respect to the sun, they will reach the moon. When the trajectory of Domingo from the earth to the moon is plotted using (5) rather than (1), the graph appears to be an arc of the earth's orbit about the sun because Q is much larger than D. Therefore, rather than plot the points generated with the sun at O, we graph the points as they would appear from earth by subtracting $G(t)$ from the point generated at time t. Figure 8(a) shows a 12-day apparent trajectory from earth when the swans fly at $c = 6.817$ LU/day leaving the earth at midnight, that is, when the sun, earth, and moon, initially in that order, are on the non-negative x-axis and the swans are at position $(Q + R, 0)$.



(a) Apparent earth-to-moon trajectory. (b) Apparent moon-to-earth trajectory.

Figure 8. Apparent trajectories.

At first glance, this trajectory may appear to be counter-intuitive since the curve goes down and to the right rather than up and to the right, as might be expected. But let's consider the first iteration. Because of the alignment of the earth and the moon, the y-coordinate of Domingo's position remains at 0. Meanwhile, the earth has moved so that its y-coordinate at time Δt is positive. It follows that the y-coordinate of Domingo's apparent position is negative rather than positive.

To implement these dynamics, we modify swan to be swan2 with four parameters, where θ is the initial polar angle of the swans' position with respect to the earth, T is a phase shift of the moon, Δt is a time step, and c is the swans' speed. For various combinations of θ and T, the swans may initially be unable to see the moon, such as when $\theta = 0$ and $\mathsf{T} = p/2$.

```
swan2[θ_,T_,Δt_, c_]:= Block[{x, y, X, Y, δ, t, bag, flag},
    x=Q+R Cos[θ]; y=R Sin[θ]; bag={{x-G1[0], y-G2[0]}}; flag=True; t=0;
    While[flag, X=μ1[t, T]−x; Y=μ2[t, T]−y;
        δ = √(X² + Y²); If[(δ <r) || (t>100), flag=False];
        x=x+X c Δt/δ; y=y+Y c Δt/δ; t=t+Δt;
        bag=Append[bag, {x-G1[t], y-G2[t]}]];
    Print["there: ",t," days"];
    ListPlot[bag, PlotJoined→True, AspectRatio→Automatic]];
```

As before, with R, r, p, Q, G_1[t], G_2[t], μ_1[t], μ_2[t] already defined, swan2[0, 0, 0.001, 6.817] generates Figure 8(a).

When Domingo returns to earth, the swans always fly toward the earth. However, unlike in the earth-moon model of the last section, now the earth is moving. A slight modification of swan2 enables us to generate return trajectories:

```
swan3[θ_,T_,Δt_, c_]:= Block[{x, y, X, Y, δ, t, bag, flag},
    x=μ1[0, T]+r Cos[θ]; y=μ2[0, T]+r Sin[θ];
    bag={{x-G1[0], y-G2[0]}}; flag=True; t=0;
    While[flag, X=G1[t]−x; Y=G2[t]−y;
        δ = Sqrt[X^2 + Y^2]; If[(δ <R) || (t>100), flag=False];
        x=x+X c Δt/δ; y=y+Y c Δt/δ; t=t+Δt;
        bag=Append[bag, {x-G1[t], y-G2[t]}]];
    Print["back: ",t," days"];
    ListPlot[bag, PlotJoined→True, AspectRatio→Automatic]];
```

The apparent trajectory to the earth given by swan3[$\pi/3, -11.27, 0.001, 6.817$] is shown in Figure 8(b), which corresponds to the earth's center being initially at $(Q, 0)$, the moon at phase $\mathsf{T} = -11.27$ days, the swans at polar angle $\theta = \pi/3$ with respect to the moon, and $c = 6.817$ LU/day. This time, Domingo's return trip lasts 8.00 days. So Godwin's guess was good, at least for this choice of parameters.

A few more pursuit questions

Here are a few problems to try.

- Another way to attack Godwin's consistency problem of 12 versus 8 is to focus on the number of swans in the wedge. In the story, two swans die on the moon, so that Domingo's wedge has 23 swans on the return trip. Develop a model between wedge speed and swans in the wedge, so that in the earth-moon model, 12 and 8 days are consistent times for the trip to the moon and back.
- In the sun-earth-moon model, for a given positive number λ, find trajectory pairs so that the ratio of flight times to the earth and the moon is λ. For a given value of c, find the optimal λ values.
- Imagine that the Little Prince's asteroid B-612 pictured in Figure 1(b) is in the asteroid belt of our solar system; plot trajectories of the Little Prince's flight to the earth and home again for various speeds c and configurations of the earth and B-612.

Acknowledgment. Copyright permission for Figure 1(b) from Harcourt.

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