EDITOR

Michael K. Kinyon
Indiana University South Bend
South Bend, IN 46634

Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Michael K. Kinyon, Indiana University South Bend, South Bend, IN 46634.

A Geometric Series from Tennis

James Sandefur (sandefur@georgetown.edu), Georgetown University, Washington, DC 20057

During the Wimbledon tennis finals, the commentator mentioned that one of the players was winning 60% of his points on serve. I began wondering what fraction of the games a person should win if the probability of winning any particular point was p. In answering this question, I used some basic probability and summed a geometric series. Others might want to share this with their students.

As a reminder, the winner of a game is the first person to score 4 points, unless the game reaches a 3-to-3 tie. Then it continues until someone goes ahead by 2. (The four points are called 15, 30, 40 and game, by the way.)

Assume that the probability that player A wins a point is p. The probability that player A wins the game within the first 6 points is the probability that A leads by a score of 3-to-0, 3-to-1, or 3-to-2, and then wins the next point, which is

$$\left[\binom{3}{0}p^3 + \binom{4}{1}p^3(1-p) + \binom{5}{2}p^3(1-p)^2\right]p = p^4(15 - 24p + 10p^2),$$

after simplification. If neither player has won by the 6th point, then the score must be tied at 3-to-3. The probability of a 3-to-3 tie is

$$\binom{6}{3}p^3(1-p)^3 = 20p^3(1-p)^3.$$

After that, the game must be won after an even number of points. The probability of winning on the 8th point is the probability of a 3-to-3 tie, followed by winning the next two points, which is

$$20p^3(1-p)^3p^2.$$

The probability of winning on the 10th point is the probability of a 3-to-3 tie, splitting the next two points, and then winning the 9th and 10th points, which is

$$20p^3(1-p)^3\big[2p(1-p)\big]p^2.$$

More generally, the probability of winning on the $(2n+6)$th point is the probability of a 3-to-3 tie, $20p^3(1-p)^3$, splitting each pair of the next $2n-2$ points $[2p(1-p)]^{n-1}$, and then winning the last 2 points p^2, which is

$$20p^3(1-p)^3\big[2p(1-p)\big]^{n-1}p^2.$$

Thus, the probability of A's winning the game is

$$f(p) = p^4(15-24p+10p^2) + 20p^3(1-p)^3p^2\sum_{n=0}^{\infty}\big[2p(1-p)\big]^n.$$

Summing the geometric series gives the function

$$f(p) = p^4(15-24p+10p^2) + \frac{20p^5(1-p)^3}{1-2p(1-p)},$$

whose graph is shown in Figure 1.

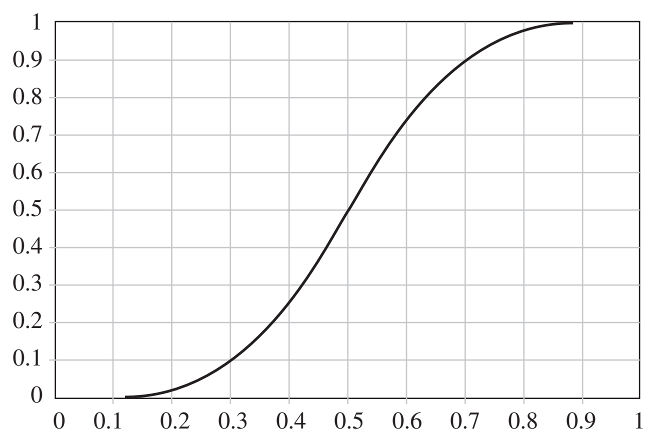


Figure 1. Graph of probability function f of winning a game as function of the probability p of winning a point.

The answer to my original question is that players who win 60% of their points on serve will win $f(0.6) \approx 0.74$ or about 74% of their service games.

The graph suggests that f has rotational symmetry about the point $(0.5, 0.5)$, although this is not apparent from the form of f. We can verify the rotational symmetry by noting that if q is the probability that the second player wins a point, then $f(q)$ gives the probability that the second player wins the game. But $p+q=1$, so $f(q) = f(1-p)$. Since one of the two players must win, $f(p) + f(1-p) = 1$. If we take $p = 0.5 + x$ and $g(x) = f(0.5+x) - 0.5$, we find that

$$g(x) + g(-x) = f(0.5+x) + f(0.5-x) - 1 = f(p) + f(1-p) - 1 = 0,$$

so $g(-x) = -g(x)$. Thus, g is an odd function, and hence f has rotational symmetry.

From the graph, it also appears that there is a point of inflection at $p = 0.5$. It would be tedious to check that $f''(0.5) = 0$. However, if we remember the derivation

$$f''(x) = \lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

and apply that to $g(x)$, we see that

$$f''(0.5) = g''(0) = \lim_{h\to 0} \frac{g(h) + g(-h) - g(0)}{h^2} = \lim_{h\to 0} \frac{0}{h^2} = 0,$$

which shows that $p = 0.5$ gives a point of inflection. This means that players gain the most in the number of games they win by increasing p when $p \approx 0.5$. In other words, if you are a weak or a strong player relative to your opponent (p small or large), then a small improvement in your serve (increasing p) doesn't result in as much improvement in the number of games you win as an improvement against a comparable opponent (p about 0.5) does. A simple computation shows that $f'(0.5) = 2.5$, so a 1% gain in p will result in about a 2.5% increase in your likelihood of winning a game.

On Sums of Cubes

Hajrudin Fejzić (hfejzic@csusb.edu), Dan Rinne (drinne@csusb.edu), and Bob Stein (bstein@csusb.edu), Department of Mathematics, California State University, San Bernardino, CA 92407

A well-known identity for the sum of the first n cubes is

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2. \tag{1}$$

Some of our students noticed that, curiously, equality still holds if $n - 1$ is replaced by 2, that is

$$1^3 + 2^3 + \cdots + (n-2)^3 + 2^3 + n^3 = (1 + 2 + \cdots + (n-2) + 2 + n)^2 .$$

This observation led us to ask whether other such switches are possible. In this note, we investigate those triples (k, m, n) for which

$$\sum_{j=1}^{k-1} j^3 + m^3 + \sum_{j=k+1}^{n} j^3 = \left(\sum_{j=1}^{k-1} j + m + \sum_{j=k+1}^{n} j \right)^2 . \tag{2}$$

Clearly (2) holds (because of (1)) if $m = k$, so in what follows we assume $m \neq k$. Furthermore, as our students pointed out, $(n-1, 2, n)$ is a solution for every $n \geq 2$. Do other solutions exist, and if so, do they fit nice patterns? Our inquiry led us to some interesting and unexpected answers, and ultimately to a connection with an unsolved problem in number theory.

Our first result gives a necessary and sufficient condition for a triple (k, m, n) to be a solution of (2). (We assume that k and m are positive integers not exceeding n.)

Theorem 1. *A triple (k, m, n) with $m \neq k$ satisfies (2) if and only if either*

(a) $k = n - 1$ *and* $m = 2$, *or*

(b) *there exists an integer $p \geq 2$ and a positive divisor s of $3p(p-1)$ for which*

$$(k, m, n) = \left(2(p-1) + \frac{3(p-1)p}{s}, 2p + s, m + k - p \right) . \tag{3}$$

Furthermore, different pairs (p, s) yield different solutions.

Proof. If we rewrite (2) as

$$\sum_{j=1}^{n} j^3 + m^3 - k^3 = \left(\sum_{j=1}^{n} j + (m-k)\right)^2.$$

and square the right side, then using (1) we deduce that

$$(m^3 - k^3) = n(n+1)(m-k) + (m-k)^2.$$

Since we assumed that $m \neq k$, it follows that (2) is equivalent to

$$m^2 + km + k^2 = n(n+1) + (m-k). \tag{4}$$

By rewriting this as $(m+k)^2 = n^2 + m\,(k+1) + n - k$, we see that since $k \leq n$, we must have $m + k > n$. Let $p = m + k - n$. Substituting $n = m + k - p$ into (4) and simplifying, we get

$$(p-1)(2m-p) = k(m-2p). \tag{5}$$

We consider two cases:

Case 1. $p = 1$. In this case, it follows from (5) that $m = 2$ and so we have the triple $(n-1, 2, n)$.

Case 2. $p \geq 2$. From (5) it follows that $m \neq 2p$ and

$$k = \frac{(p-1)(2m-p)}{m-2p}.$$

Suppose $m - 2p < 0$. Then $2m - p < 0$ since $k > 0$. Thus $2m < p = m + k - n$ so $n < k - m$, which is a contradiction since $1 \leq m, k \leq n$. Hence $m > 2p$. We rewrite k as

$$k = 2(p-1) + \frac{3p(p-1)}{m-2p}.$$

From this we see that $s = m - 2p$ divides $3p(p-1)$ since k is an integer. Therefore, for $p \geq 2$, (k, m, n) satisfies (4) if and only if

$$k = 2(p-1) + \frac{3p(p-1)}{s}, \quad m = 2p + s, \quad \text{and} \quad n = m + k - p$$

for some positive s of $3p(p-1)$.

Finally, from $p = m + k - n$ we see that different values of p produce different triples (k, m, n) and if p is fixed, then different divisors s of $3p(p-1)$ yield different values of m. ■

We now look further at solutions of type (b) in the theorem, which states that whenever there are positive integers p, s, and t with $p \geq 2$ and $st = 3p(p-1)$, there is always a solution:

$$m = 2p + s, \quad k = 2(p-1) + t, \quad \text{and} \quad n = 3p + s + t - 2. \tag{6}$$

Notice that, from these expressions, it follows that for a given value of m or k (as well as n, for which it's obvious), there are only finitely many solutions. Also, it must be that $m \geq 5$ and $k \geq 3$ in order for a solution to exist. On the other hand, we observe that for any given $m \geq 5$, there will always be a solution (take $s = 1$ or 2, for example). Of course the same holds for any $k \geq 3$. So an interesting question is what can be said about the solutions of (2) if n is fixed. The following theorem partially answers this question. We omit the somewhat complex proof.

Theorem 2. *If $n^2 + n + 1 \neq q$ and $n^2 + n + 1 \neq 3q$, where $q > 3$ is prime, then there are corresponding unequal k and $m \neq 2$ satisfying (2). Otherwise the only solutions are (m, m, n) and $(n - 1, 2, n)$.*

Whether or not there are infinitely many n such that $n^2 + n + 1 = q$ or $n^2 + n + 1 = 3q$, where $q > 3$ is prime is unknown.

Symmetry at Infinity

Richard Winton (winton@tarleton.edu) Tarleton State University, Stephenville, TX 76402

In a calculus class dealing with applications of integrals, we encountered the center of mass/centroid problems for plane regions. Working out examples in class, we computed the centroid of the region of finite area bounded by $y = x^2$ and $y = \sqrt[3]{x}$. At the conclusion of the problem, Jason, a chemistry major, asked the following question.

> If the two given functions had been $y = x^2$ and $y = \sqrt{x}$, would the centroid have been located at the point $(\frac{1}{2}, \frac{1}{2})$?

Since the graphs of $y = x^2$ and $y = \sqrt{x}$ intersect at the points (0, 0) and (1, 1) and are symmetric about the line $y = x$, the question seems like a reasonable one. However, rather than simply answer the question for $y = x^2$ and $y = \sqrt{x}$, I replied by asking the same question concerning any pair of functions of the form $y = x^n$ and $y = \sqrt[n]{x}$, where n is a positive integer, since they possess the same points of intersection and symmetry. This paper is devoted to answering that question.

Thus for each positive integer n, define $f_n(x) = x^n$ and $f_n^{-1}(x) = \sqrt[n]{x} = x^{1/n}$. For $n = 1$, $f_1(x) = x = f_1^{-1}(x)$, so that no area is bounded by $f_1(x)$ and $f_1^{-1}(x)$. However, for $n > 1$, $f_n(x)$ and $f_n^{-1}(x)$ bound a region R_n of finite area between (0, 0) and (1, 1). We proceed now to derive a general formula for the coordinates (x_n, y_n) of the centroid C_n of R_n in terms of n ($n > 1$).

Since any invertible function and its inverse are symmetric about the line $y = x$, we have $x_n = y_n$. (See Figure 1.) Thus a single function of n provides the coordinates (x_n, y_n) of C_n. At first glance, many students conjecture that the region R_n is also symmetric about the line $y = 1 - x$. (See Figure 2.) Based on the assumption of this additional symmetry, the students conclude that R_n must have centroid $C_n(\frac{1}{2}, \frac{1}{2})$ at the intersection of $y = x$ and $y = 1 - x$.

However, if $f(x) \geq g(x)$ for $a \leq x \leq b$ and R is a plane lamina of constant density ρ whose area is bounded by $f(x)$ and $g(x)$ over the interval $[a, b]$, then the mass and first moments of R are

$$\mu = \rho \int_a^b \left[f(x) - g(x)\right] dx,$$

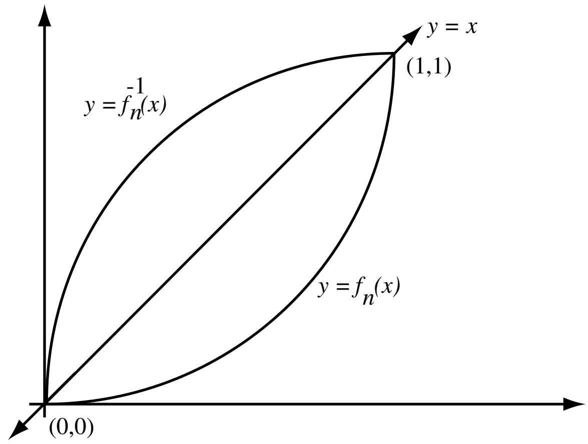


Figure 1.

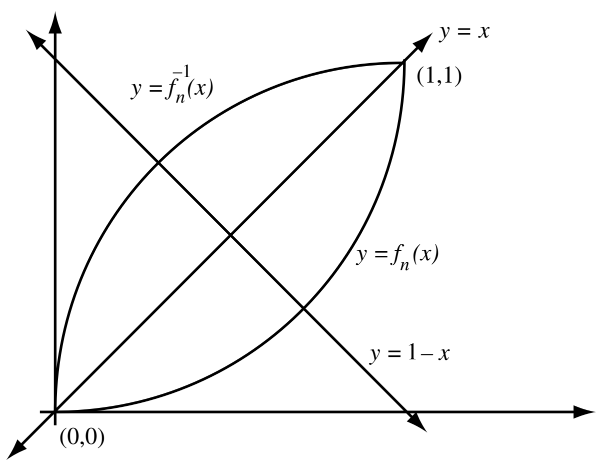


Figure 2.

$$M_x = \frac{\rho}{2} \int_a^b \left(\left[f(x) \right]^2 - \left[g(x) \right]^2 \right) dx,$$

and

$$M_y = \rho \int_a^b x \left[f(x) - g(x) \right] dx,$$

respectively. Thus the centroid of R is $C(x, y)$, where

$$x = \frac{M_y}{\mu} = \frac{\int_a^b x[f(x) - g(x)]dx}{\int_a^b [f(x) - g(x)]dx} \quad \text{and} \quad y = \frac{M_x}{\mu} = \frac{\frac{1}{2}\int_a^b ([f(x)]^2 - [g(x)]^2)dx}{\int_a^b [f(x) - g(x)]dx}.$$

Since $f_n^{-1}(x) = x^{1/n} \geq x^n = f_n(x)$ for $n > 1$ and $0 \leq x \leq 1$, then

$$\int_0^1 \left(x^{1/n} - x^n \right) dx = \frac{n}{n+1} \cdot x^{(n+1)/n} - \frac{1}{n+1} \cdot x^{n+1} \bigg|_0^1 = \frac{n-1}{n+1}$$

and

$$\int_0^1 x\left(x^{1/n} - x^n\right) dx = \frac{n}{2n+1} \cdot x^{(2n+1)/n} - \frac{1}{n+2} \cdot x^{n+2}\Big|_0^1 = \frac{(n+1)(n-1)}{(n+2)(2n+1)}.$$

Thus

$$x_n = \frac{\int_0^1 x(f_n^{-1}(x) - f_n(x))\,dx}{\int_0^1 (f_n^{-1}(x) - f_n(x))\,dx} = \frac{(n+1)(n-1)}{(n+2)(2n+1)} \div \frac{n-1}{n+1} = \frac{n^2+2n+1}{2n^2+5n+2} = y_n.$$

Hence for $n > 1$,

$$C_n = \left(\frac{n^2+2n+1}{2n^2+5n+2}, \frac{n^2+2n+1}{2n^2+5n+2}\right).$$

For example, the centroid of R_2 is $C_2(\frac{9}{20}, \frac{9}{20})$, while the centroid of R_3 is $C_3(\frac{16}{35}, \frac{16}{35})$. In fact, for $n > 1$, $C_n \neq (\frac{1}{2}, \frac{1}{2})$. Upon closer inspection, the graphs of $y = f_n(x)$ and $y = f_n^{-1}(x)$ reveal that R_n is somewhat wider toward the origin than the opposite end, resulting in the centroid C_n being located nearer $(0, 0)$ than $(1, 1)$. Figures 3, 4, and 5 provide graphs of $y = f_n(x)$ and $y = f_n^{-1}(x)$ for $n = 2$, 7, and 30. The resulting region R_n and centroid C_n are displayed on each plot.

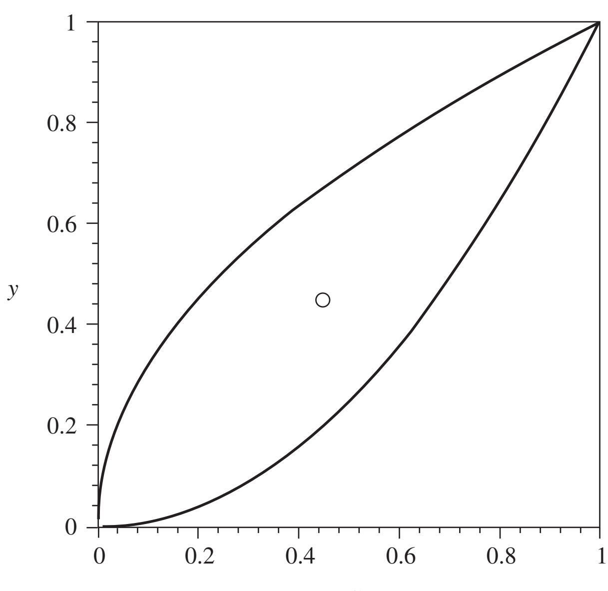


Figure 3. $n = 2$

As n gets large, R_n expands toward the perimeter of the square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$, and C_n approaches the center of the square at the point $(\frac{1}{2}, \frac{1}{2})$. More specifically,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2+5n+2} = \frac{1}{2},$$

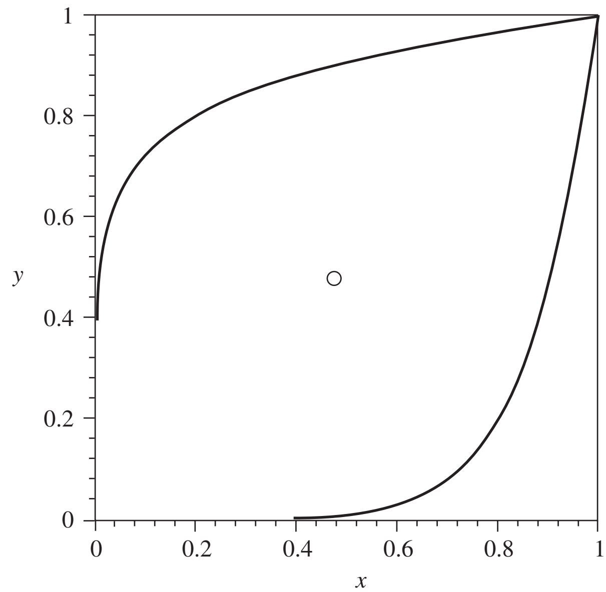


Figure 4. $n = 7$

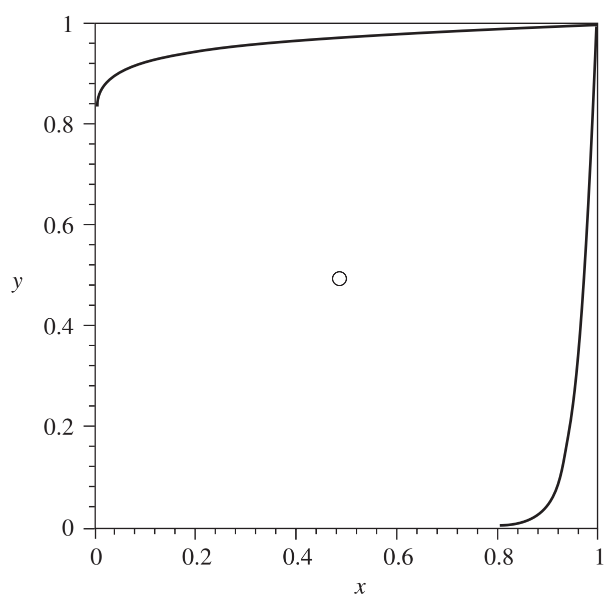


Figure 5. $n = 30$

and so $\lim_{n\to\infty} C_n = (\frac{1}{2}, \frac{1}{2})$. Hence the initial conjectures that R_n is symmetric about the line $y = 1 - x$ and has centroid $C_n = (\frac{1}{2}, \frac{1}{2})$ are actually valid only in the limiting case for C_∞. In other words, relative to the line $y = 1 - x$, and in terms of the parameter n, R_n finally achieves "symmetry at infinity."

The Flip-Side of a Lagrange Multiplier Problem

Angelo Segalla (asegalla@csulb.edu) and Saleem Watson (saleem@csulb.edu), California State University, Long Beach, Long Beach, California 90840

Introduction. A typical optimization problem in beginning calculus courses is the 'fencing-a-field' problem:

> *Find the dimensions of the rectangular field of maximum area for a fixed perimeter.*

There is a natural "flip-side" to this problem:

> *Find the dimensions of the rectangular field of minimum perimeter for a fixed area.*

It is apparent that these problems are related, but what, exactly, is the relationship between them? Do other optimization problems have a flip-side? If so, how does one formulate the flip-side of a given problem?

We give an answer to these questions by considering the more general problem of optimizing a function f of two variables subject to a constraint $g(x, y) = c$ using Lagrange multipliers. As the fencing-a-field problem suggests, the flip-side of a problem involves interchanging the roles of f and g (a process that is meaningful because the Lagrange multiplier condition $\nabla f = \lambda \nabla g$ is symmetric in f and g). In this note we define what is meant by the flip-side of a problem and prove a result that relates an extremum of a problem to an extremum of its flip-side. In following the steps of the proof, students can see how properties of the gradient—in particular the property that the gradient points in the direction of the greatest rate of increase in the values of a function—can be useful visual tools in analyzing optimization problems.

Several articles on Lagrange multipliers have appeared in the CMJ (see for instance [**1**], [**2**], [**3**], [**5**]), but it seems that the general relationship between a problem and its flip-side (as we call it here) has not been discussed.

The general problem. To better see the relationship between a problem and its flip-side, let's solve a specific fencing-a-field problem. Suppose the amount of fencing available is 40 units, say. Then the problem is this: Maximize $A(x, y) = xy$ subject to the constraint $P(x, y) = 2x + 2y = 40$. The answer is a square of side 10 and area 100. The flip-side problem is *about fields of area* 100: Minimize P subject to the constraint $A(x, y) = 100$. Again the answer is a square of side 10.

Following this example leads us to the following general situation. Suppose f and g are functions of two variables and f has a local maximum (or minimum) value $m = f(a, b)$ at the point (a, b) subject to the constraint $g(x, y) = c$. The *flip-side* problem is: Does g have a local extremum at (a, b) on the constraint $f(x, y) = m$? And if so, is the extremum a local maximum or minimum?

We show that in general (under appropriate smoothness conditions on f and g) the flip-side problem always has a local extremum at (a, b), and the type of extremum depends on whether ∇f and ∇g point in the same or opposite directions at (a, b). We will say that f has a *local maximum point* at (a, b) on the constraint $g(x, y) = c$ if $f(a, b) > f(x, y)$ for all (x, y) on the level set $g(x, y) = c$ in some disc centered at (a, b).

Theorem. *Suppose f and g are smooth functions of two variables, $\nabla f(a,b) \neq 0$, and*

$$\nabla f(a,b) = \lambda \nabla g(a,b).$$

Let $f(a,b) = c_1$, $g(a,b) = c_2$. If f has a local maximum (minimum) point at (a,b) on the constraint $g(x,y) = c_2$, then the following hold:

1. *If $\lambda > 0$, then $g(x,y)$ has a local minimum (maximum) at (a,b) on the constraint $f(x,y) = c_1$.*
2. *If $\lambda < 0$, then $g(x,y)$ has a local maximum (minimum) at (a,b) on the constraint $f(x,y) = c_1$.*

Proof. We prove the result when f has a local maximum at (a,b). Since

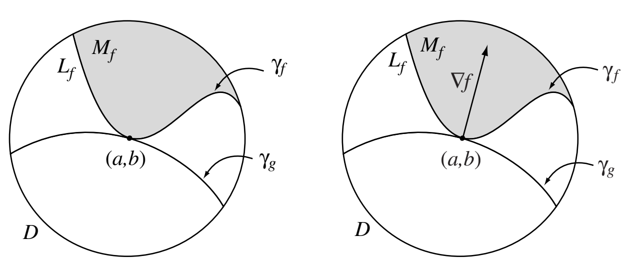
$$\nabla f(a,b) \neq 0 \quad \text{and} \quad \nabla g(a,b) \neq 0$$

the level sets $f(x,y) = c_1$ and $g(x,y) = c_2$ are smooth curves with nonvanishing tangent vectors in some disc centered at (a,b) [**1**]. Let's call these curves γ_f and γ_g, respectively, as in Figure 1(a). With this terminology, the hypothesis states that f has a local maximum value $c_1 = f(a,b)$ on γ_g. So we can find a small enough disc D on which γ_f and γ_g are smooth curves and $f(x,y) < c_1$ at all other points of γ_g inside D.

Now $f(x,y) > c_1$ on one side of γ_f and $f(x,y) < c_1$ on the other side in D. This is because if $f(x,y) > c_1$ on both sides, then $c_1 = f(a,b)$ is the minimum value of f in D, and so $\nabla f(a,b) = 0$, contradicting the hypothesis. Thus the intersection of the sets

$$M_f = \{(x,y) : f(x,y) \geq c_1\} \quad \text{and} \quad L_f = \{(x,y) : f(x,y) \leq c_1\}$$

in D is γ_f as in Figure 1(a). We have used the letters M and L to indicate the sets where f is "more than" and "less than" (or equal to) c_1, respectively. Of course g has the same properties as f, so in the same way we define M_g and L_g with respect to c_2.

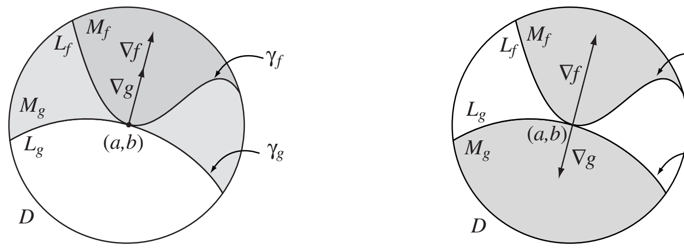


(a) γ_g is contained in L_f. (b) ∇f points into M_f.

Figure 1. f is larger than c_1 on one side of γ_f and smaller on the other.

Since f attains its maximum value on γ_g at (a,b), it follows that the values of f on γ_g are all less than or equal to c_1, that is, $\gamma_g \subset L_f$. See Figure 1(a). (This implies that γ_g is on one side of γ_f, so these curves do not cross in D.) Because the gradient always points in the direction of greatest increase, it follows that $\nabla f\ (a,b)$ points into M_f (Figure 1(b)).

1. If $\lambda > 0$ then $\nabla f\ (a,b)$ and $\nabla g(a,b)$ point in the same direction, so we must have $M_f \subset M_g$ as in Figure 2(a). In this case $\gamma_f \subset M_g$, that is, the values of g



(a) ∇f, ∇g point in same direction. (b) ∇f, ∇g point in opposite directions.

Figure 2. ∇g tells us on which side of γ_g is M_g

on γ_f are all greater than or equal to c_2. Thus g has the local minimum value $g(a, b) = c_2$ on γ_f.

2. If $\lambda < 0$ then $\nabla f\ (a, b)$ and $\nabla g(a, b)$ point in opposite directions. So we must have $M_g \subset L_f$ as in Figure 2(b). In this case $\gamma_f \subset L_g$, that is, the values of g on γ_f are all less than or equal to c_2. Thus g has the local maximum value $g(a, b) = c_2$ on γ_f. ∎

Moving a river and other flip-side problems. Once a constrained optimization problem has been solved, we can use the theorem to state and solve the flip-side problem. For an applied problem, it is interesting to consider the physical interpretation of the flip-side. For the fencing-the-field problem we want to maximize *area*, for the flip-side we want to minimize *perimeter*. We consider two other common first-semester calculus problems.

The milkmaid problem [**2**] asks for the minimum distance a milkmaid needs to walk from her home to fetch water from a river and take it to the barn. Specifically, suppose her home is at $(-3, 0)$, the barn at $(3, 0)$, and the river is the line $R(x, y) = 100$, where $R(x, y) = 16x + 15y$. To walk to a point (x, y) and then to the barn the maid travels a distance $d(x, y) = \sqrt{(x+3)^2 + y^2} + \sqrt{(x-3)^2 + y^2}$. The problem can now be stated as follows: Minimize d subject to the constraint $R(x, y) = 100$. Using Lagrange multipliers we find that $\lambda > 0$ and the minimum is $d(4, \frac{12}{5}) = 10$. By our theorem the flip-side problem is to maximize R subject to the constraint $d(x, y) = 10$; moreover, the local maximum value is $R(4, \frac{12}{5}) = 100$. We can interpret the flip-side problem as follows: If the maid insists that she will walk a distance of *exactly* 10, then we must move the river for her! That is, we must find the maximum value of c so that she can *just* reach the river $R(x, y) = c$ and then walk to the barn, travelling a total distance of 10.

Another ubiquitous problem in first semester calculus courses is the ladder-around-the-corner problem: Find the length of the longest ladder that can go around a rectangular corner with hallways of fixed widths [**3**]. Using our theorem it is easy to state and solve the flip-side of this problem, but what physical quantity are we actually minimizing in the flip-side problem?

In general, every problem of the type we describe here has a flip-side. For applied problems it is interesting to try and find meaning for the quantities being optimized in the flip-side problem.

References

1. H. R. Bailey, ‘Hidden’ boundaries in constrained max-min problems, *College Math. J.* **22** (1991) 227–229.
2. W. Barnier and D. Martin, Unifying a family of extrema problems, *College Math. J.* **28** (1997) 388–391.

3. P. Driscoll and W. P. Fox, Presenting the Kuhn-Tucker conditions using a geometric approach, *College Math. J.* **27** (1996) 101–108.
4. J. Marsden, A. Tromba, and A. Weinstein, *Basic Multivariable Calculus*, Springer Verlag, 1993.
5. J. Nunemacher, Lagrange multipliers can fail to determine extrema, *College Math. J.* **34** (2003) 60–62.
6. A. Shenk, *Calculus and Analytic Geometry*, 3rd ed., Scott, Foresman, 1984.
7. J. Stewart, *Calculus: Early Transcendentals*, 5th ed., Brooks/Cole, 2003.

Another Proof for the p-series Test

Yang Hansheng (hsyang921@yahoo.com.cn), Southwest University of Science and Technology, Mianyang, China 621002, and Bin Lu (binlu@csus.edu), California State University Sacramento, Sacramento, CA 95819

It is well known that the p-series is $1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$ converges for $p > 1$ and diverges for $p \leq 1$. In standard calculus textbooks (such as [**3**] and [**4**]), this is usually shown using the integral test. In this note, we provide an alternative proof of the convergence of the p-series without using the integral test. In fact, our proof is an extension of the nice result given by Cohen and Knight [**2**].

We begin by giving the following estimate for the partial sum of a p-series:

Lemma. *Let $s_n(p)$ be the nth partial sum of the p-series $\sum_{k=1}^{\infty} 1/k^p$.*

(a) *For $p > 0$,*

$$1 - \frac{1}{2^p} + \frac{2}{2^p}s_n(p) < s_{2n}(p) < 1 + \frac{2}{2^p}s_n(p),$$

(b) *For $p < 0$,*

$$1 + \frac{2}{2^p}s_n(p) < s_{2n}(p) < 1 - \frac{1}{2^p} + \frac{2}{2^p}s_n(p).$$

Proof. As $s_n(p)$ is the nth partial sum,

$$\begin{aligned} s_{2n}(p) &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2n)^p} \\ &= 1 + \left[\frac{1}{2^p} + \frac{1}{4^p} + \cdots + \frac{1}{(2n)^p}\right] + \left[\frac{1}{3^p} + \frac{1}{5^p} + \cdots + \frac{1}{(2n-1)^p}\right]. \end{aligned}$$

For $p > 0$,

$$s_{2n}(p) > 1 + \frac{1}{2^p}s_n(p) + \left[\frac{1}{4^p} + \frac{1}{6^p} + \cdots + \frac{1}{(2n)^p}\right].$$

Thus,

$$s_{2n}(p) > 1 + \frac{1}{2^p}s_n(p) - \frac{1}{2^p} + \frac{1}{2^p}s_n(p) = 1 - \frac{1}{2^p} + \frac{2}{2^p}s_n(p).$$

Also,

$$s_{2n}(p) < 1 + \frac{1}{2^p}s_n(p) + \left[\frac{1}{2^p} + \frac{1}{4^p} + \cdots + \frac{1}{(2n)^p}\right] = \frac{2}{2^p}s_n(p) + 1.$$

This proves (1). We can prove (2) in a similar manner. ■

From these estimates, we have the following test for the p-series:

Theorem. *The p-series is divergent when $p \leq 1$, and in this case,*

$$\lim_{n\to\infty} \frac{s_{2n}(p)}{s_n(p)} = \frac{2}{2^p}. \tag{1}$$

The p-series is convergent for $p > 1$, and in this case,

$$\frac{2^p - 1}{2^p - 2} \leq \lim_{n\to\infty} s_n(p) \leq \frac{2^p}{2^p - 2}. \tag{2}$$

Proof. When $p < 0$, the p-series is divergent since the general term does not converge to 0. So we consider $0 < p \leq 1$. Assume that the p-series is convergent, that is, $\lim_{n\to\infty} s_n(p) = s(p)$. From the lemma we obtain the following inequality by letting $n \to \infty$:

$$1 - \frac{1}{2^p} + \frac{2}{2^p}s(p) = \frac{2^p - 1}{2^p} + \frac{2}{2^p}s(p) \leq s(p),$$

and from this inequality we have

$$0 < \frac{2^p - 1}{2^p} \leq \frac{2^p - 2}{2^p}s(p) \leq 0,$$

which is a contradiction. Thus the p-series is divergent when $p \leq 1$. We obtain

$$\lim_{n\to\infty} \frac{s_{2n}(p)}{s_n(p)} = \frac{2}{2^p}$$

by dividing both inequalities of the lemma by $s_n(p)$ and letting $n \to \infty$. This proves (1).

Now let $p > 1$. From the inequality of the first part of the lemma, we have

$$s_n(p) < s_{2n}(p) < 1 + \frac{2}{2^p}s_n(p),$$

and then $0 < (1 - (2/2^p))s_n(p) < 1$. Hence, $s_n(p) < 2^p/(2^p - 2)$ for all n, so the sequence $\{s_n(p)\}$ is bounded. Furthermore, it is increasing, so the limit $\lim_{n\to\infty} s_n(p)$ exists, and hence the p-series is convergent for $p > 1$. The inequality is obtained by letting $n \to \infty$ in the first part of the lemma. ■

Remarks. (1) From the theorem, for $p > 1$, we obtain an estimate for the sum of a p-series. For example, when $p = 2$, we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.65.$$

The theorem gives the estimate $1.5 \leq \lim_{n\to\infty} s_n(p) \leq 2$.

(2) The theorem also provides a way of calculating some interesting limits related to the p-series. For example, consider the p-series $\sum_{k=1}^{\infty} 1/k^p$ with $p = 1/3$. It is divergent and $\lim_{n\to\infty} s_n(p) = \infty$. Then from the first part of the theorem, we can calculate the limit:

$$\lim_{n\to\infty} \frac{1 + \frac{1}{\sqrt[3]{2}} + \cdots + \frac{1}{\sqrt[3]{n}} + \cdots + \frac{1}{\sqrt[3]{2n}}}{1 + \frac{1}{\sqrt[3]{2}} + \cdots + \frac{1}{\sqrt[3]{n}}} = \sqrt[3]{4}.$$

Similarly for $p = 1$,

$$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{2n}}{1 + \frac{1}{2} + \cdots + \frac{1}{n}} = 1.$$

References

1. R. P. Boas, Jr. and J. W. Wrench, Jr., Partial sum of the harmonic series, *Amer. Math. Monthly* **78** (1971) 864–870.
2. T. Cohen and W. J. Knight, Convergence and divergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$, *Math. Mag.* **52** (1979) 178.
3. D. Hughes-Hallet, *et al.*, *Calculus: Single Variable*, 3rd ed., Wiley, 2002.
4. J. Stewart, *Calculus: Early Transcendentals*, 5th ed., Brooks/Cole, 2003.

Taylor Series—A Matter of Life or Death

Mathematics can even be a matter of life or death. During the Russian revolution, the mathematical physicist Igor Tamm was seized by anti-communist vigilantes at a village near Odessa where he had gone to barter for food. They suspected he was an anti-Ukrainian communist agitator and dragged him off to their leader.

Asked what he did for a living, he said that he was a mathematician. The sceptical gang-leader began to finger the bullets and grenades slung around his neck. "All right," he said, "calculate the error when the Taylor series approximation of a function is truncated after n terms. Do this and you will go free; fail and you will be shot." Tamm slowly calculated the answer in the dust with his quivering finger. When he had finished, the bandit cast his eye over the answer and waved him on his way.

Tamm won the 1958 Nobel prize for Physics but he never did discover the identity of the unusual bandit leader. But he found a sure way to concentrate his students' minds on the practical importance of Mathematics!

The Observer (U.K.)
16 May 1993