Do Dogs Know Bifurcations?

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Elvis burst upon the mathematical scene in May, 2003. The second author's article "Do Dogs Know Calculus?" [**2**] introduced his dog Elvis and Elvis's ability to solve a classic optimization problem. Peruchet and Gallego's article "Do Dogs Know Related Rates Rather Than Optimization?" [**4**] gave an alternative explanation of how dogs (including their own) might solve the problem. Elvis's surprising repudiation of that explanation in [**3**] inspired this article. Here, we explore Elvis's problem-solving ability when he must choose between two qualitatively different options. Such a situation induces a bifurcation in his optimal strategy. As a bonus, our analysis reveals a neat geometric proof of the arithmetic mean–geometric mean inequality.

In the original problem, Elvis is on the shoreline and wants to retrieve a ball thrown x meters into the water and z meters downshore, as in Figure 1. Elvis runs along the shore at speed r m/s to a point y meters upshore from the ball, then swims to the ball at speed s m/s.

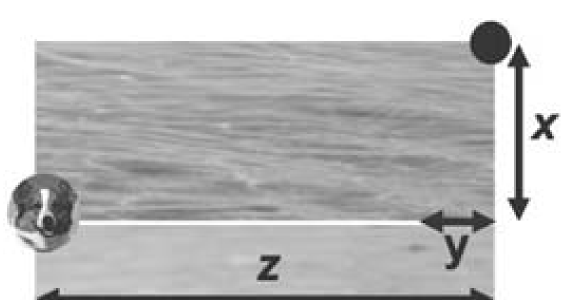

Figure 1. The original problem.

It is shown in [**2**] that the total time to the ball is minimized with

$$\hat{y} = \frac{x}{\sqrt{(r/s)^2 - 1}} \tag{1}$$

if $z > \hat{y}$. This solution is remarkable because the optimal entry point is independent of the distance z. Also, the distance $\hat{y}$ is a linear function of x. In [**2**], Elvis's actual entry points for a large number of throws are presented. The scatter plot of these points shows a remarkable linear trend that closely matches the line of optimal entry points. Thus, it seems that Elvis is able to solve this general problem.

In [**4**], Perruchet and Gallego start with the function $d(t)$ giving the distance between Elvis and the ball. As he runs along the shoreline, the rate of change $d'(t)$ is negative and increasing. The position at which $|d'(t)|$ reaches the swim speed s (that is, $d'(t) = -s$) is shown to be exactly the optimal $\hat{y}$ in (1) above. This allows a different interpretation of how Elvis gets to the ball. Instead of some internal calculation of x (how far out in the water the ball is) and then y (where to enter the water), perhaps Elvis runs along the shore until he senses that he could make better progress to the ball by swimming. That is, instead of solving a global optimization problem, perhaps Elvis is solving a local related rates problem.

Fortunately, Elvis has provided more clues about his problem-solving strategy. As noted in [**3**], when Elvis starts *in* the water and a ball is thrown a long distance parallel to the shore, he first swims to shore, then runs along the shore, and finally swims back out to the ball. Thus, at least in this situation, Elvis is apparently viewing the task globally. However, his behavior raises three new questions.

First of all, what are the possible optimal paths? This question is easily answered. For any path, Elvis either reaches the shore or stays in the water. If he stays in the water, then swimming directly to the ball will result in the shortest time. If he reaches the shore, then the optimal path will involve swimming and running in straight lines. Thus, the optimal path will either be a straight swim to the ball (designated S), or a path (designated SRS) consisting of three straight lines. If the ball is thrown a short distance, S will be faster than SRS. The longer the throw, the more likely that SRS will be the quicker path. The second question is, what is the bifurcation point at which the optimal strategy changes from S to SRS? And third, does Elvis change his strategy at the optimal point? That is, does Elvis know bifurcations?

The swim-run-swim problem

Given the discussion above, we compare the times of the S and SRS paths. To find the optimal SRS path, suppose Elvis starts x_1 meters out in the water and races to a ball that is z meters downshore and x_2 meters out into the water, as in Figure 2. He first

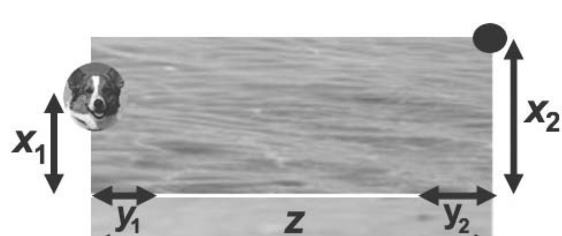

Figure 2. The SRS problem.

swims ashore with speed s m/s to a point y_1 meters downshore, then runs along the beach at speed r m/s to a point y_2 meters upshore from the ball, and finally swims out to the ball.

The total time to reach the ball is given by

$$T = \frac{\sqrt{x_1^2 + y_1^2}}{s} + \frac{z - y_1 - y_2}{r} + \frac{\sqrt{x_2^2 + y_2^2}}{s}. \tag{2}$$

If we consider T as a function of y_1 and y_2, it reaches a minimum when both partial derivatives are zero. We have

$$\frac{\partial T}{\partial y_i} = \frac{y_i}{s\sqrt{x_i^2 + y_i^2}} - \frac{1}{r}$$

for $i = 1, 2$. Setting each partial derivative equal to 0 and eliminating $\frac{1}{r}$ gives

$$\frac{y_1}{\sqrt{x_1^2 + y_1^2}} = \frac{y_2}{\sqrt{x_2^2 + y_2^2}}$$

which are the cosines of the angles in Figure 3.

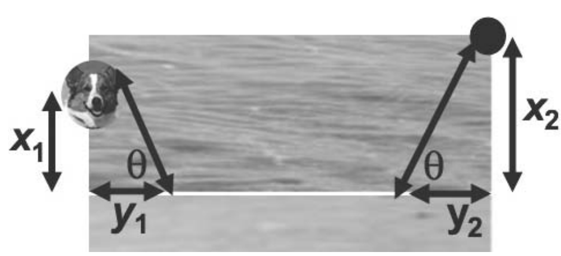

Figure 3. Equal angles.

We conclude that **angle in equals angle out**! Further, solving $\partial T/\partial y_i = 0$ for y_i gives

$$\hat{y}_i = \frac{x_i}{\sqrt{(r/s)^2 - 1}} \tag{3}$$

for $i = 1, 2$. The value for $\hat{y}_2$ coincides exactly with the solution of the original problem, if $z > \hat{y}_1 + \hat{y}_2$.

Upon reflection, these results are obvious. This is often the case when such a nice result emerges. Think of the problem in two parts. Step (i) is to go from a point in the water to a distant point on the shore. Step (ii) is to go from that point on the shore to another distant point in the water. Since the original solution (1) is independent of z, steps (i) and (ii) are independent. In fact, step (ii) is simply the original problem. Further, the two steps are equivalent with step (i) being step (ii) covered in reverse. Therefore, the angles must be equal. Thought of in a different way, since (1) is independent of z, choose $z = \hat{y}_1 + \hat{y}_2$ so that there is no running at all. Then, analogous to light reflecting off a mirror or a billiard ball bouncing off a rail, the optimal path has angle in equal to angle out.

This suggests a possible explanation of Elvis's behavior. Perhaps Elvis uses a small set of rules. For example,

1. If the ball is close, swim directly to it. (Elvis does this.)
2. If the ball is far away, then (A) get out of the water and (B) solve the shore to ball problem.

The experience gained solving (2B) can help in (2A), since angle in equals angle out. The correct angle might "feel" right. Notice that this leaves open the question of how Elvis actually solves (2B). Such an explanation is consistent with artificial life models such as Craig Reynolds's boids [**5**] and with constructal theory [**1**].

Bifurcation points

The next step is to compare the S and SRS strategies. Substituting (3) into (2) gives the total time for the optimal SRS path, which can be written in the form

$$T_{\text{SRS}} = \frac{z}{r} + \frac{x_1 + x_2}{sr/\sqrt{r^2 - s^2}}.$$

The time to swim directly to the ball is given by

$$T_{\text{S}} = \frac{\sqrt{z^2 + (x_2 - x_1)^2}}{s}.$$

We want to find all values of z for which $T_{\text{SRS}} = T_{\text{S}}$. Squaring the equation $T_{\text{SRS}} = T_{\text{S}}$, using the quadratic formula to solve for z, and discarding the extraneous solution gives the critical value

$$\tilde{z} = \frac{x_1 + x_2 + 2\frac{r}{s}\sqrt{x_1 x_2}}{\sqrt{(r/s)^2 - 1}}. \tag{4}$$

For $z < \tilde{z}$, the fastest route is to swim directly to the ball. For $z > \tilde{z}$, the fastest route is the SRS path found above. The value $\tilde{z}$ is called a bifurcation point, since the nature of the optimal solution changes at this value.

This result leads to some interesting insights. First, there is no bifurcation point if $s > r$. If swimming is faster than running, then swimming directly to the ball is always optimal. Second, if $s < r$ and $s \approx r$, then $\tilde{z}$ is very large. For shorter distances z, the small advantage that running provides cannot compensate for having to swim the extra distance to shore.

Finally, as $\frac{r}{s} \to \infty$ in equation (4), $\tilde{z} \to 2\sqrt{x_1 x_2}$. For the physical problem with large values of $\frac{r}{s}$, the optimal strategy is to swim the shortest distance possible getting to shore and back to the ball. That is, as $\frac{r}{s} \to \infty$, the shape of the optimal SRS path will form three sides of a trapezoid, as in Figure 4.

The physical problem also helps us determine the length of the top of this trapezoid. As $\frac{r}{s} \to \infty$, the running time along the beach approaches zero, so the total SRS time equals the time to swim $x_1 + x_2$ meters. At the bifurcation point, the S and SRS times are equal, so the S path must also have length $x_1 + x_2$ meters, as in Figure 5. Check this out geometrically!

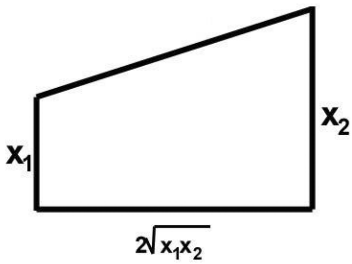

Figure 4. The SRS path.

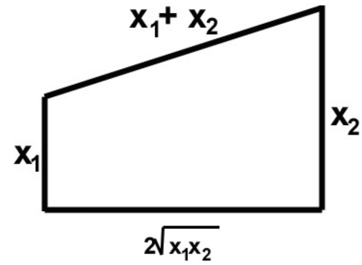

Figure 5. A mean triangle.

Figure 5 shows us that given two positive numbers, x_1 and x_2, $\sqrt{x_1 x_2} \leq \frac{1}{2}(x_1 + x_2)$. Thus, in the process of analyzing optimal retrieval strategies, we have discovered a picture proof for the relationship between the geometric mean and arithmetic mean of two numbers. The figure also reveals that equality holds only when $x_1 = x_2$. How interesting that by thinking rather intuitively about a problem in the physical world, we discover truths about the mathematical world.

In the case that a ball is thrown parallel to the shore, $x_2 = x_1$. Here, the equation $T_{\text{SRS}} = T_{\text{S}}$ simplifies and the bifurcation point is

$$\tilde{z} = 2x\sqrt{\frac{r/s + 1}{r/s - 1}}. \tag{5}$$

The bifurcation experiment

Since Elvis already revealed his willingness to bifurcate [**3**], the remaining question is whether he bifurcates at the correct point. To answer this, the second author and two undergraduate research students took Elvis to the same Lake Michigan beach where the first experiment [**2**] was done. Taking the average of several timed trials, Elvis's running speed was estimated to be 6.39 m/s and his swimming speed to be 0.73 m/s. However, once Elvis actually started chasing the ball, his running speed slowed down considerably to an average of 3.02 m/s. (This reduced speed was likely a combination of being tired from swimming to shore and just enjoying a lazy July afternoon.) Using equation (5), we see that the optimal bifurcation point is then $\tilde{z} = 2.56x$.

The second author stood 4 meters out in the water with Elvis and threw a ball various distances, but landing about 4 meters from the shore. One student measured the distance of the throw and the other recorded Elvis's choice. The results are in Table 1.

The data suggest several conclusions about Elvis's choices. First, it seems that there is a bifurcation. He consistently applies the SRS strategy for longer distances and swims directly to the ball for shorter ones. Secondly, there may or may not be a well-defined bifurcation point. If there is, it exists somewhere between 14 m and 15 m for this example, but without doing many more trials to narrow down the point and to show consistency, one cannot be sure. Lastly, if there is a well-defined bifurcation point, it is not where it should be. According to equation (5), the bifurcation should be at 10.24 m. Elvis's bifurcation distance was (a disappointing) 4 m larger. Thus it might be concluded that Elvis knows bifurcations qualitatively, but not quantitatively.

Lest we be too hard on Elvis, though, it should be remembered that dogs, like all of us, learn from experience and that bifurcations by their very nature make learning

Table 1.

Trial number	z (m)	Strategy
1	16.5	SRS
2	9.5	S
3	14.2	S
4	15.1	SRS
5	15	SRS
6	12	S
7	7.8	S
8	11.6	S
9	18.2	SRS

difficult. Bifurcations force a choice upon us, and we often do not have the opportunity to go back and try both options. But, in forcing the choice, bifurcations add interest to mathematics and richness to life. As the poet Robert Frost wrote,

> "Two roads diverged in a wood, and I—
> I took the one less traveled by,
> And that has made all the difference."

References

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3. Timothy J. Pennings, Response, *College Math. J.* **37** (2006) 19.
4. Pierre Perruchet and Jorge Gallego, Do dogs know related rates rather than optimization?, *College Math. J.* **37** (2006) 16–18.
5. Craig Reynolds, Flocks, herds and schools: a distributed behavioral model, *Computer Graphics: Proceedings of SIGGRAPH 87* (1987) 25–34.

Paul Halmos on Writing a Paper

In an interview, Paul Halmos was asked, "How many drafts do you usually prepare before you feel that you're there?" Halmos replied, "That's an unanswerable question because there isn't something called Draft 1 and then Draft 2 and then Draft 3. There is something called Draft 1 all right except I prefer to call it Draft Zero. And then I change a sentence, and then I change a paragraph, then I change a page, then I have to change two pages, and it's unclear when it becomes a different draft. Every single word that I publish I write at least six times."

—*CMJ*, January 2004, p. 8.