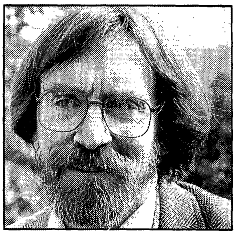
Euler and the Fundamental Theorem of Algebra

William Dunham



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Over the years, Dunham's interests have shifted toward mathematics history. Twice he has directed summer seminars for the National Endowment for the Humanities, and in 1990 he authored *Journey through Genius: The Great Theorems of Mathematics*. A second book, jointly written with wife and colleague Penny Dunham, is in the works.

A watershed event for all students of mathematics is the first course in basic high school algebra. In my case, this provided an initial look at graphs, inequalities, the quadratic formula, and many other critical ideas. Somewhere near the term's end, as I remember, our teacher mentioned what sounded like the most important result of them all—the fundamental theorem of algebra. Anything with a name like that, I figured, must be (for want of a better term) *fundamental*. Unfortunately, the teacher informed us that this theorem was much too advanced to state, let alone to investigate, at our current level of mathematical development.

Fine. I was willing to wait. However, second year algebra came and went, yet the fundamental theorem occupied only an obscure footnote from which I learned that it had something to do with factoring polynomials and solving polynomial equations. My semester in college algebra/precalculus the following year went a bit further, and I emerged vaguely aware that the fundamental theorem of algebra said that nth-degree polynomials could be factored into n (possibly complex) linear factors, and thus nth-degree polynomial equations must have n (possibly complex and possibly repeated) solutions. Of course, to that point we had done little with complex numbers and less with complex solutions of polynomial equations, so the whole business remained obscure and mysterious. Even in those pre-Watergate days, I began to sense that the mathematical establishment was engaged in some kind of cover-up to keep us ignorant of the true state of algebraic affairs.

"Oh well," I thought, "I'm off to college, where surely I'll get the whole story." Four years later I was still waiting. My undergraduate mathematics training—particularly courses in linear and abstract algebra—examined such concepts as groupoids, eigenvalues, and integral domains, but none of my algebra professors so much as mentioned the fundamental theorem. This was very unsatisfactory—a bit like reading *Moby Dick* and never encountering the whale. The cover-up had continued through college, and algebra's superstar theorem was as obscure as ever.

It was finally in a graduate school course on complex analysis that I saw a proof of this key result, and I immediately realized the trouble: the theorem really is a monster to prove in full generality, for it requires some sophisticated preliminary results about complex functions. Clearly a complete proof *is* beyond the reach of elementary mathematics.

So what does a faculty member do if an inquiring student seeks information about the fundamental theorem of algebra? It is hopeless to try to prove the thing for any precalculus student whose I.Q. lies on this side of Newton's; on the other hand, it would more or less continue the cover-up to avoid answering the question —to treat an inquiry about the fundamental theorem of algebra as though the student had asked something truly improper, delicate, or controversial—like a question about one's religion, or one's sex life, or even one's choice of personal computer.

Let me, then, suggest an intermediate option—something less rigorous than a grad school proof, yet something more satisfying than simply telling our inquisitive student to get lost. My suggestion is that we look back to the history of mathematics and to the work of that most remarkable of eighteenth century mathematicians, Leonhard Euler (1707–1783). With Euler's attempted proof of the fundamental theorem of algebra from 1749, we find yet another example of the history of mathematics serving as a helpful ingredient in the successful teaching of the subject. The reasoning is not impossibly difficult; it raises some interesting questions for further discussion; and while his is not a complete proof by any means, it does establish the result for low degree polynomials and suggests to students that this sweeping theorem is indeed reasonable.

Before addressing the subject further, we state the theorem in its modern form:

> Any nth-degree polynomial with complex coefficients can be factored into n complex linear factors.

That is,

> If $P(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_2 z^2 + c_1 z + c_0$, where $c_n, c_{n-1}, \ldots, c_2, c_1, c_0$ are complex numbers, then there exist complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that
>
> $$P(z) = c_n(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n).$$

It may come as a surprise that, to mathematicians of the mid-eighteenth century, the fundamental theorem appeared in the following guise:

> Any polynomial with real coefficients can be factored into the product of real linear and/or real quadratic factors.

Note that there is no mention here of complex numbers, either as the polynomial's coefficients nor as parts of its factors. For mathematicians of the day, the theorem described a phenomenon about *real* polynomials and their *real* factors.

As an example, consider the factorization

$$3x^4 + 5x^3 + 10x^2 + 20x - 8 = (3x - 1)(x + 2)(x^2 + 4).$$

Here the quartic has been shattered into the product of two linear fragments and one irreducible quadratic one, and all polynomials in sight are real. The theorem stated that such a factorization was possible for any real polynomial, no matter its degree.

Anticipating a bit, we see that we can further factor the quadratic expression—provided we allow ourselves the luxury of complex numbers. That is,

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$
$$= a\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

factors the real quadratic ax^2+bx+c into two, albeit rather unsightly, linear pieces. Of course, there is no guarantee these linear factors are composed of *real* numbers, for if $b^2-4ac<0$, we venture into the realm of imaginaries. In the specific example cited above, for instance, we get the complete factorization:

$$3x^4+5x^3+10x^2+20x-8 = (3x-1)(x+2)(x-2i)(x+2i).$$

This is "complete" in the sense that the real fourth-degree polynomial with which we began has been factored into the product of four *linear* complex factors, certainly as far as any factorization can hope to proceed.

It was the Frenchman Jean d'Alembert (1717–1783) who gave this theorem its first serious treatment in 1746 [**5**, p. 99]. Interestingly, for d'Alembert and his contemporaries the result had importance beyond the realm of algebra: its implications extended to the relatively new subject of calculus and in particular to the integration technique we now know as "partial fractions." As an illustration, suppose we sought the indefinite integral

$$\int \frac{28x^3-4x^2+69x-14}{3x^4+5x^3+10x^2+20x-8}\,dx.$$

To be sure, this looks like absolute agony, as all calculus teachers will readily agree. (One would have trouble finding it in the Table of Integrals of a calculus book's inside cover, unless the book is very thorough or its cover is very large.) This problem even gives a good workout to symbolic manipulators such as *Mathematica* (which required 50 seconds to find the antiderivative on my Mac II) and which were not available to eighteenth century mathematicians in any case.

But if, as d'Alembert claimed, the denominator could be decomposed into real linear and/or real quadratic factors, then the difficulties drop away. Here, the integrand becomes

$$\int \frac{28x^3-4x^2+69x-14}{(3x-1)(x+2)(x^2+4)}\,dx.$$

We then determine its partial fraction decomposition, getting

$$\int \frac{28x^3-4x^2+69x-14}{3x^4+5x^3+10x^2+20x-8}\,dx$$
$$= \int \frac{28x^3-4x^2+69x-14}{(3x-1)(x+2)(x^2+4)}\,dx$$
$$= \int \frac{1}{3x-1}\,dx + \int \frac{7}{x+2}\,dx + \int \frac{2x-3}{x^2+4}\,dx$$
$$= \tfrac{1}{3}\ln|3x-1| + 7\ln|x+2| + \ln(x^2+4) - \tfrac{3}{2}\tan^{-1}(x/2) + C,$$

and the antiderivative is found.

Thus, if the fundamental theorem were proved in general, we could conclude that for any $P(x)/Q(x)$ where P and Q are real polynomials, the indefinite integral $\int(P(x)/Q(x))\,dx$ would exist as a combination of fairly simple functions (at least theoretically). That is, we could first perform long division to reduce this rational expression to one where the degree of the numerator was less than the degree of $Q(x)$; next we consider $Q(x)$ as the product of real linear and/or real quadratic factors; then apply the partial fraction technique to break the integral into pieces of the form

$$\int \frac{A}{(ax+b)^n}\,dx \quad \text{and/or} \quad \int \frac{Bx+C}{(ax^2+bx+c)^n}\,dx;$$

and finally determine these indefinite integrals using nothing worse than natural logarithms, inverse tangents, or trigonometric substitution. Admittedly, the fundamental theorem gives no process for finding the denominator's explicit factors; but, just as the theorem guarantees the *existence* of such a factorization, so too will the *existence* of simple antiderivatives for any rational function be established.

Unfortunately, d'Alembert's 1746 attempt to prove his theorem was unsuccessful, for the difficulties it presented were simply too great for him to overcome (see [**4**, pp. 196–198]). In spite of this failure, the fundamental theorem of algebra has come to be known as "d'Alembert's Theorem" (especially in France). Attaching his name to this result may seem a bit generous, given that he failed to prove it. This is a bit like designating the Battle of Waterloo as "Napoleon's Victory."

So matters stood when Euler turned his awesome mathematical powers to the problem. At the time he picked up the scent, there was not even universal agreement that the theorem was true. In 1742, for instance, Nicholas Bernoulli had expressed to Euler his conviction that the real quartic polynomial

$$x^4-4x^3+2x^2+4x+4$$

cannot be factored into the product of real linear and/or real quadratic factors in any fashion whatever [**1**, pp. 82–83]. If Bernoulli were correct, the game was over; the fundamental theorem of algebra would have been instantly disproved.

However, Bernoulli's skepticism was unfounded, for Euler factored the quartic into the product of the quadratics

$$x^2-\left(2+\sqrt{4+2\sqrt{7}}\right)x+\left(1+\sqrt{4+2\sqrt{7}}+\sqrt{7}\right) \quad \text{and}$$
$$x^2-\left(2-\sqrt{4+2\sqrt{7}}\right)x+\left(1-\sqrt{4+2\sqrt{7}}+\sqrt{7}\right).$$

Those with a taste for multiplying polynomials can check that these complicated factors yield the fairly innocent quartic above; far more challenging, of course, is to figure out how Euler derived this factorization in the first place. (Hint: it was not by guessing.)

By 1742, Euler claimed he had proved the fundamental theorem of algebra for real polynomials up through the sixth-degree [**3**, p. 598], and in a landmark 1749 article titled "Recherches sur les racines imaginaires des équations" [**1**, pp. 78–169], he presented his proof of the general result which we shall now examine (see also [**5**, pp. 100–102]). We stress again that his argument failed in its ultimate mission. That is, Euler furnished only a partial proof which, in its full generality, suffered logical shortcomings. Nonetheless, even with these shortcomings, one cannot fail to recognize the deftness of a master at work.

He began with an attack on the quartic:

Theorem. *Any quartic polynomial* $x^4 + Ax^3 + Bx^2 + Cx + D$ *where* A, B, C, *and* D *are real can be decomposed into two real factors of the second degree.*

Proof. Euler first observed that the substitution $x = y - (A/4)$ reduces the original quartic into one lacking a cubic term—a so-called "depressed quartic." Depressing an nth-degree polynomial by a clever substitution that eliminates its $(n-1)$st-degree term is a technique whose origin can be traced to the sixteenth century Italian mathematician Gerolamo Cardano in his successful attack on the cubic equation [**3**, p. 265].

With this substitution, the quartic becomes

$$\left(y - \frac{A}{4}\right)^4 + A\left(y - \frac{A}{4}\right)^3 + B\left(y - \frac{A}{4}\right)^2 + C\left(y - \frac{A}{4}\right) + D,$$

and the only two sources of a y^3 term are

$$\left(y - \frac{A}{4}\right)^4 = y^4 - Ay^3 + \cdots \quad \text{and} \quad A\left(y - \frac{A}{4}\right)^3 = A(y^3 - \cdots) = Ay^3 - \cdots.$$

Upon simplifying, we find that the "y^3" terms cancel and there remains the promised depressed quartic in y.

Not surprisingly, there are advantages to factoring a depressed quartic rather than a full-blown one; yet it is crucial to recognize that any factorization of the depressed quartic yields a corresponding factorization of the original. For instance, suppose we were trying to factor $x^4 + 4x^3 - 9x^2 - 16x + 20$ into a product of two quadratics. The substitution $x = y - \frac{4}{4} = y - 1$ depresses this to $y^4 - 15y^2 + 10y + 24$, and a quick check confirms the factorization:

$$y^4 - 15y^2 + 10y + 24 = (y^2 - y - 2)(y^2 + y - 12).$$

Then, making the reverse substitution $y = x + 1$ yields

$$x^4 + 4x^3 - 9x^2 - 16x + 20 = (x^2 + x - 2)(x^2 + 3x - 10),$$

and the original quartic is factored as claimed.

Having reduced the problem to that of factoring depressed quartics, Euler noted that we need only consider $x^4 + Bx^2 + Cx + D$, where B, C, and D are real.

At this point, two cases present themselves:

Case 1. $C = 0$.

This amounts to having a depressed quartic $x^4 + Bx^2 + D$, which is just a quadratic in x^2. (Euler omitted discussion of this possibility, perhaps because it could be handled in two fairly easy subcases by purely algebraic means.)

First of all, suppose $B^2 - 4D \geq 0$ and apply the quadratic formula to get the decomposition into two second-degree *real* factors as follows:

$$x^4 + Bx^2 + D = \left[x^2 + \frac{B - \sqrt{B^2 - 4D}}{2}\right]\left[x^2 + \frac{B + \sqrt{B^2 - 4D}}{2}\right].$$

For instance, $x^4 + x^2 - 12 = (x^2 - 3)(x^2 + 4)$.

Less direct is the case where we try to factor $x^4 + Bx^2 + D$ under the condition that $B^2 - 4D < 0$. The previous decomposition no longer works, since the factors containing $\sqrt{B^2 - 4D}$ are not real. Fortunately, a bit of algebra shows that the quartic can be written as the difference of squares and thus factored into quadratics as follows:

$$\begin{aligned} x^4 + Bx^2 + D &= \left[x^2 + \sqrt{D}\right]^2 - \left[x\sqrt{2\sqrt{D} - B}\right]^2 \\ &= \left[x^2 + \sqrt{D} - x\sqrt{2\sqrt{D} - B}\right]\left[x^2 + \sqrt{D} + x\sqrt{2\sqrt{D} - B}\right]. \end{aligned}$$

A few points must be made about this factorization. First, $B^2 - 4D < 0$ implies that $4D > B^2 \geq 0$, and so the expression $\sqrt{D}$ in the preceding factorization is indeed real. Likewise, $4D > B^2$ guarantees that $\sqrt{4D} > \sqrt{B^2}$, or simply $2\sqrt{D} > |B| \geq B$, and so the expression $\sqrt{2\sqrt{D} - B}$ is likewise real. In short, the factors above are two real quadratics, as we hoped.

For example, when factoring $x^4 + x^2 + 4$, we find $B^2 - 4D = -15 < 0$ and the formula yields $x^4 + x^2 + 4 = [x^2 - x\sqrt{3} + 2][x^2 + x\sqrt{3} + 2]$.

Case 2. $C \neq 0$.

Here Euler observed that a factorization of his depressed quartic into real quadratics—if it exists—*must* take the form

$$x^4 + Bx^2 + Cx + D = (x^2 + ux + \alpha)(x^2 - ux + \beta) \tag{1}$$

for some real numbers u, α, and β yet to be determined. Of course, this form is necessary since the "ux" in one factor must have a compensating "$-ux$" in the other.

Euler multiplied out the right-hand side of (1) to get:

$$x^4 + Bx^2 + Cx + D = x^4 + (\alpha + \beta - u^2)x^2 + (\beta u - \alpha u)x + \alpha\beta,$$

and then equated coefficients from the first and last of these expressions to generate three equations:

$$B = \alpha + \beta - u^2, \qquad C = \beta u - \alpha u = (\beta - \alpha)u, \quad \text{and} \quad D = \alpha\beta.$$

Note that B, C, and D are just the coefficients of the original polynomial, whereas u, α, and β are unknown real numbers whose *existence* Euler had to establish.

From the first two of these we conclude that

$$\alpha + \beta = B + u^2 \quad \text{and} \quad \beta - \alpha = \frac{C}{u}.$$

It may be worth noting that since $0 \neq C = (\beta - \alpha)u$, then u itself is nonzero, so its presence in the denominator above is no cause for alarm.

If we both add and subtract these two equations, we arrive at

$$2\beta = B + u^2 + \frac{C}{u} \quad \text{and} \quad 2\alpha = B + u^2 - \frac{C}{u}. \tag{2}$$

Euler recalled that $D=\alpha\beta$ and consequently:

$$4D = 4\alpha\beta = (2\beta)(2\alpha) = \left(B + u^2 + \frac{C}{u}\right)\left(B + u^2 - \frac{C}{u}\right).$$

In other words, $4D = u^4 + 2Bu^2 + B^2 - (C^2/u^2)$, and multiplying through by u^2 gives us

$$u^6 + 2Bu^4 + (B^2 - 4D)u^2 - C^2 = 0. \tag{3}$$

It may appear that things have gotten worse, not better, for we have traded a fourth-degree equation in x for a sixth-degree equation in u. Admittedly, (3) is also a cubic in u^2, so we can properly conclude that there is a real solution for u^2; this, unfortunately, does not guarantee the existence of a *real* value for u, which was Euler's objective.

Undeterred, he noticed four critical properties of (3):

(a) B, C, and D are known, so the only unknown here is u.

(b) B, C, and D are real.

(c) the polynomial is even and thus its graph is symmetric about the y-axis.

(d) the constant term of this sixth-degree polynomial is $-C^2$.

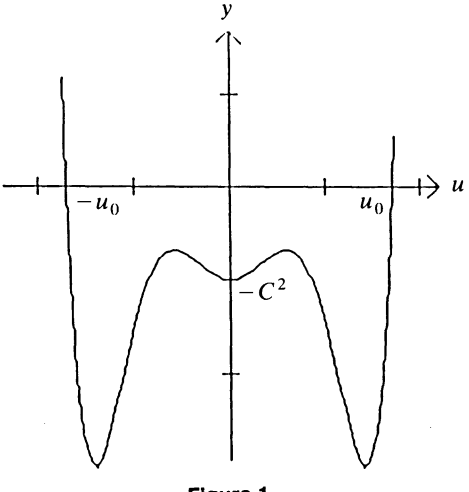

Figure 1

$y = u^6 + 2Bu^4 + (B^2 - 4D)u^2 - C^2$

Here Euler's mathematical agility becomes especially evident. He was considering a sixth-degree real polynomial whose graph looks something like that shown in Figure 1. This has a negative y-intercept at $(0, -C^2)$ since C is a nonzero real number. Additionally, since the polynomial is monic of even degree, its graph climbs toward $+\infty$ as u becomes unbounded in either the positive or negative direction. By a result from analysis we now call the intermediate value theorem—but which Euler took as intuitively clear—we are guaranteed the *existence* of real numbers $u_0 > 0$ and $-u_0 < 0$ satisfying this sixth-degree equation.

Using the positive solution u_0 and returning to equations in (2), Euler solved for β and α, getting real solutions

$$\beta_0 = \tfrac{1}{2}\left(B + u_0^2 + \frac{C}{u_0}\right) \quad \text{and} \quad \alpha_0 = \tfrac{1}{2}\left(B + u_0^2 - \frac{C}{u_0}\right)$$

and, since $u_0 > 0$, these fractions are well-defined.

In summary, under the case that $C \neq 0$, Euler had established the existence of real numbers u_0, α_0, and β_0 such that

$$x^4 + Bx^2 + Cx + D = \left(x^2 + u_0 x + \alpha_0\right)\left(x^2 - u_0 x + \beta_0\right).$$

We thus see that any depressed quartic with real coefficients—and by extension any real quartic at all—does have a factorization into two real quadratics, whether or not $C = 0$. Q.E.D.

At this point, Euler immediately observed, "...it is also evident that any equation of the fifth degree is also resolvable into three real factors of which one is linear and two are quadratic" [**1**, p. 95]. His reasoning was simple (see Figure 2). Any *odd-degree* polynomial—and thus any fifth-degree polynomial $P(x)$—is guaranteed by the intermediate value theorem to have at least one real x-intercept, say at $x = a$. We then write $P(x) = (x - a)Q(x)$, where $Q(x)$ is a polynomial of the fourth-degree, and the previous result allows us to decompose $Q(x)$, in turn, into two real quadratic factors.

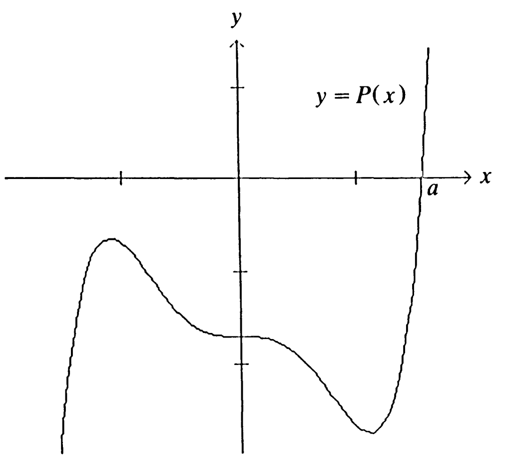

Figure 2

By now, a general strategy was brewing in his mind. He realized that *if* he could prove his decomposition for real polynomials of degree $4, 8, 16, 32$, and in general of degree 2^n, then he could prove it for any real polynomials whatever.

Why is this? Suppose, for instance, we were trying to establish that the polynomial

$$x^{12} - 3x^9 + 52x^8 + 3x^3 - 2x + 17$$

could be factored into real linear and/or real quadratic factors. We would simply multiply it by x^4 to get

$$x^{16} - 3x^{13} + 52x^{12} + 3x^7 - 2x^5 + 17x^4.$$

Assuming that Euler had proved the 16th-degree case, he would know that this latter polynomial would have such a factorization, obviously containing the four linear factors x, x, x, and x. If we merely cancelled them out, we would of necessity be left with the real linear and/or real quadratic factors for the original 12th-degree polynomial.

And so, with typical Eulerian cleverness, he reduced the entire issue to a few simpler cases. Having disposed of the fourth-degree case, he next claimed, "Any equation of the eighth degree is always resolvable into two real factors of the fourth degree" [**1**, p. 99]. Since each of the fourth-degree factors was itself decomposable into a pair of real quadratics, which themselves can be broken into (possibly complex) linear factors, he would have succeeded in shattering the eighth-degree polynomial into eight linear pieces. From there he went to the 16th-degree before finally tackling the general situation, namely showing that any real polynomial of degree 2^n can be factored into two real polynomials each of degree 2^{n-1} [**1**, p. 105].

It was a brilliant strategy. Unfortunately, the proofs he furnished left something to be desired. As we shall see, for the higher-degree cases the arguments became hopelessly complicated, and his assertions as to the existence of *real* numbers satisfying certain equations were unconvincing. Consider, for instance, the eighth-degree case. It began in a fashion quite similar to its fourth-degree counterpart, namely by first depressing the octic and imagining that it has been factored into the two quartics:

$$\begin{aligned} &x^8 + Bx^6 + Cx^5 + Dx^4 + Ex^3 + Fx^2 + Gx + H \\ &\quad = (x^4 + ux^3 + \alpha x^2 + \beta x + \gamma)(x^4 - ux^3 + \delta x^2 + \varepsilon x + \phi). \qquad (4) \end{aligned}$$

One multiplies the quartics, equates the resulting coefficients with the known quantities $B, C, D, \ldots$ to get seven equations in seven unknowns, and asserts that there exist real values of $u, \alpha, \beta, \gamma, \ldots$ satisfying this system.

The parallels with what he had previously done are evident. But what made this case so much less successful was Euler's admission that for equations of higher degree, "...it will be very difficult and even impossible to find the equation by which the unknown u is determined" [**1**, p. 97]. In short, he was unwilling or unable to solve this system explicitly for u.

Ever resourceful, Euler decided to look again at the depressed quartic in (1) for inspiration. As it turned out, an entirely different line of reasoning suggested itself, a line that he thought could be extended naturally to the eighth and higher-degree cases:

Assuming that the quartic in (1) has four roots p, q, r, and s, Euler wrote:

$$\begin{aligned} (x^2 + ux + \alpha)(x^2 - ux + \beta) &= x^4 + Bx^2 + Cx + D \\ &= (x-p)(x-q)(x-r)(x-s), \qquad (5) \end{aligned}$$

and from this factorization he drew three key conclusions.

First, upon multiplying the four linear factors on the right of (5), we see immediately that the coefficient of x^3 is $-(p+q+r+s)$; hence $p+q+r+s=0$ since the quartic is depressed.

Second, the quadratic factor $(x^2 - ux + \beta)$ must arise as the product of two of the four linear factors. Thus, $(x^2 - ux + \beta)$ could be $(x-p)(x-r) = x^2 - (p+r)x + pr$; it could just as well be $(x-q)(x-r) = x^2 - (q+r)x + qr$; and so on. This implies that, in the first case, $u = p + r$, whereas in the second $u = q + r$. In fact, it is clear that u can take any of the $\binom{4}{2} = 6$ values

$$\begin{aligned} R_1 &= p + q \qquad & R_4 &= r + s \\ R_2 &= p + r & R_5 &= q + s \\ R_3 &= p + s & R_6 &= q + r. \end{aligned}$$

Since u is an unknown having these six possible values, it must be determined by the sixth-degree polynomial $(u - R_1)(u - R_2)(u - R_3)(u - R_4)(u - R_5)(u - R_6)$. This conclusion, of course, is entirely consistent with the explicit sixth-degree polynomial for u that Euler had found in (3).

But Euler made one additional observation. Because $p + q + r + s = 0$, it follows that $R_4 = -R_1$, $R_5 = -R_2$, and $R_6 = -R_3$. Hence the sixth-degree polynomial becomes

$$\begin{aligned} &(u - R_1)(u + R_1)(u - R_2)(u + R_2)(u - R_3)(u + R_3) \\ &\quad = (u^2 - R_1^2)(u^2 - R_2^2)(u^2 - R_3^2). \end{aligned}$$

The constant term here—which is to say, this polynomial's y-intercept—is simply $-R_1^2R_2^2R_3^2 = -(R_1R_2R_3)^2$. This constant, Euler stated, was a negative real number, again in complete agreement with his conclusions from equation (3).

To summarize, Euler had provided an entirely different argument to establish that, in the quartic case, u is determined by a $\binom{4}{2} =$ 6th-degree polynomial with a negative y-intercept. This was the critical conclusion he had already drawn, but here he drew it without *explicitly* finding the equation determining u.

The advantage of this alternate proof for the quartic case was that it could be used to analyze the depressed octic in (4). *Assuming* that the octic was decomposed into eight linear factors, Euler mimicked his reasoning above to deduce that for each different combination of four of these eight factors, we would get a different value of u. Thus, u would be determined by a polynomial of degree $\binom{8}{4} = 70$ having a negative y-intercept. He then confidently applied the intermediate value theorem to get his desired *real* root u_0, and from this he claimed that the other real numbers α_0, β_0, γ_0, δ_0, ε_0, and ϕ_0 exist as well.

Euler reasoned similarly in the 16th-degree case, claiming that "...the equation which determines the values of the unknown u will necessarily be of the 12870th degree" [**1**, p. 103]. The degree of this (obviously unspecified) equation is simply $\binom{16}{8} = 12870$, as his pattern suggested. By this time, Euler's comment that it was "...very difficult and even impossible..." to specify these polynomials had become something of an understatement.

From there it was a short and entirely analogous step to the general case: that any real polynomial of degree 2^n could be factored into two real polynomials of degree 2^{n-1}. With that, his proof was finished.

Or was it? Unfortunately, his analyses of the 8th-degree, 16th-degree, and general cases were flawed and left significant questions unanswered. For instance, if we look back at the quartic in (5), how could Euler assert that it has four roots? How could he assert that the octic in (4) has eight?

More significantly, what is the nature of these supposed roots? Are they real? Are they complex? Or are they an unspecified—and perhaps entirely unimagined—new kind of number? If so, can they be added and multiplied in the usual fashion?

These are not trivial questions. In the quartic case above, for example, if we are uncertain about the nature of the roots p, q, r, and s, then we are equally uncertain about the nature of their sums R_1, R_2, R_3. Consequently, there is no guarantee whatever that mysterious expressions such as $-(R_1R_2R_3)^2$ are negative real numbers. But if these y-intercepts are not negative reals, then the intermediate value arguments that Euler applied to the 8th-degree, 16th-degree, and general cases fall apart completely.

It appears, then, that Euler had started down a very promising path in his quest of the fundamental theorem. His first proof worked nicely in dealing with fourth- and fifth-degree real polynomials. But as he pursued this elusive theorem deeper into the thicket, complications involving the existence of his desired real factors became overwhelming. In a certain sense, he lost his way among the enormously high degree polynomials that beckoned him on, and his general proof vanished in the wilderness.

So even Euler suffered setbacks, a fact from which comfort may be drawn by lesser mathematicians (a category that includes virtually everybody else in history). Yet, before the dust settles and his attempted proof is consigned to the scrap heap, I think it deserves at least a modest round of applause, for it certainly bears signs of his characteristic cleverness, boldness, and mental agility as he leaps between the polynomial's analytic and algebraic properties. More to the point, the fourth- and fifth-degree arguments are understandable by good precalculus students and can give them not only a deeper look at this remarkable theorem but also a glimpse of a mathematical giant at work. For even when he stumbled, Leonhard Euler left behind signs of great insight. Such, perhaps, is the mark of genius.

EPILOGUE

The fundamental theorem of algebra—the result that established the complex numbers as the optimum realm for factoring polynomials or solving polynomial equations—thus remained in a very precarious state. D'Alembert had not proved it; Euler had given an unsatisfactory proof. It was obviously in need of major attention to resolve its validity once and for all.

Such a resolution awaited the last year of the eighteenth century and came at the hands of one of history's most talented and revered mathematicians. It was the 22-year old German Carl Friedrich Gauss (1777–1855) who first presented a reasonably complete proof of the fundamental theorem (see [**4**, p. 196] for an interesting twist on this oft-repeated statement). Gauss' argument appeared in his 1799 doctoral dissertation with the long and descriptive title, "A New Proof of the Theorem That Every Integral Rational Algebraic Function [i.e., every polynomial with real coefficients] Can Be Decomposed into Real Factors of the First or Second Degree" (see [**5**, pp. 115–122]). He began by reviewing past attempts at proof and giving criticisms of each. When addressing Euler's "proof," Gauss raised the issues cited above, designating Euler's mysterious, hypothesized roots as "shadowy." To Gauss, Euler's attempt lacked "...the clarity which is required in mathematics" [**2**, p. 491]. This clarity he attempted to provide, not only in the dissertation but in two additional proofs from 1816 and another from 1848. As indicated by his return to this result throughout his illustrious career, Gauss viewed the fundamental theorem of algebra as a great and worthy project indeed.

We noted previously that this crucial proposition is seen today in somewhat greater generality than in the early nineteenth century, for we now transfer the theorem entirely into the realm of complex numbers in this sense: the polynomial with which we begin no longer is required to have real coefficients. In general, we consider nth-degree polynomials having complex coefficients, such as

$$z^7 + 6iz^6 - (2+i)z^2 + 19.$$

In spite of this apparent increase in difficulty, the fundamental theorem nonetheless proves that it can be factored into the product of (in this case seven) linear terms having, of course, complex coefficients. Interestingly, modern proofs of this result almost never appear in algebra courses. Rather, today's proofs rest upon a study of the *calculus* of complex numbers and thus move quickly into the realm of genuinely advanced mathematics (just as my high school algebra teacher had so truthfully said).

And so, we reach the end of our story, a story that can be a valuable tale for us and our students. It addresses an oft-neglected theorem of much importance; it allows the likes of Jean d'Alembert, Leonhard Euler, and Carl Friedrich Gauss to cross the stage; and it gives an intimate sense of the historical development of great mathematics in the hands of great mathematicians.

References

1. Leonhard Euler, Recherches sur les racines imaginaires des équations, *Memoires de l'académie des sciences de Berlin* (5) (1749), 1751, 222–288 (*Opera Omnia*, (1) 6, 78–147).
2. J. Fauvel and J. Gray, *The History of Mathematics: A Reader*, The Open University, London, 1987.
3. Morris Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1972.
4. John Stillwell, *Mathematics and its History*, Springer-Verlag, New York, 1989.
5. Dirk Struik (Ed.), *A Source Book in Mathematics: 1200–1800*, Princeton University Press, 1986.

Early Burn Out?

..."Have you spoken to any spatial geometricians?"

"I did better than that. I called in a neighbor's kid who used to be able to solve Rubik's cube in seventeen seconds. He sat on the step and stared at it for over an hour before pronouncing it irrevocably stuck. Admittedly he's a few years older now and has found out about girls, but it's got me puzzled."

Douglas Adams, *Dirk Gently's Holistic Detective Agency*, Pocket Books, 1987.