# To Build a Better Box

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Most calculus students have encountered the problem of finding the maximum volume of a box that is constructed from a rectangular piece of cardboard by cutting equal squares from each corner and folding up the sides. Have you ever asked your students to actually construct such a box? I have. The students soon discover that the most practical part of this "application of calculus" is the fact that it opens the door to more practical methods of construction. To begin with, removing the corners is ridiculous. If you just cut along one side of each square and use the squares to reinforce the sides, the result is a much stronger box. Another thing they notice, with a little gentle persuasion, is that a box without a top is not very useful. This observation gives me a chance to suggest the construction method shown in Figure 1.

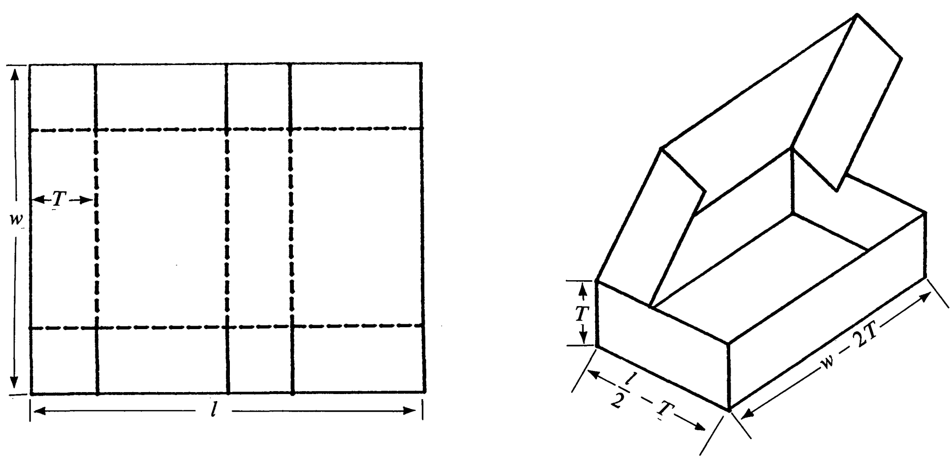



Figure 1.

If you cut along the solid lines and fold along the dotted lines, four well-placed staples will secure a fairly useable box. My students have dubbed this one the *Pizza Box*.

With the no-top construction, cutting out $T$ by $T$ squares from the corners of a rectangle with length $l$ and width $w$ ($w \leqslant l$), the volume is given by

$$V(T) = T(l - 2T)(w - 2T)$$

for $T < w/2$. With the Pizza Box construction, the volume is given by

$$V(T) = T(l/2 - T)(w - 2T) = T(l - 2T)(w - 2T)/2 \qquad \text{for} \quad T < w/2.$$

Therefore, for any value of $T$, the volume is half as large using the Pizza Box method, and the maximum occurs at the same value of $T$ in each case.

With a little more prodding, some students will come to the conclusion that restricting the shape of the rectangular piece of cardboard limits the maximum volume of the box. They also can see that this is not a reasonable "real world" restriction. To allow variable dimensions for the rectangle and variable corner sizes would usually require the calculus of several variables. Since this is not available to students when I want to cover this topic, I suggest the following approach.

Suppose $A$ square inches of cardboard is used to construct a box using the Pizza method. Fixing the height at $T$ inches, find the dimensions of the rectangle that will maximize the box's volume.

Taking $w = A/l$ in Figure 1, we have

$$V(l) = T(\tfrac{l}{2} - T)(\tfrac{A}{l} - 2T).$$

Then $V'(l) = 0$ when $l = \sqrt{A}$. The cardboard's required dimensions are therefore $l = w = \sqrt{A}$. Using the $\sqrt{A}$ by $\sqrt{A}$ cardboard, we want to find the height $T$ that will maximize this volume. Thus, we begin with

$$V(T) = T\left(\tfrac{1}{2}\sqrt{A} - T\right)\left(\sqrt{A} - 2T\right).$$

Then $V'(T) = 0$ for $T = \sqrt{A}/6$ and the maximum volume of the Pizza Box is $V_{\text{pizza}} = A^{3/2}/27$.

For classroom development, use $A = 144$ square inches because it gives a nice maximum volume of 64 cubic inches when $l = w = 12$ and $T = 2$.

By the time we have solved the Pizza Box problem, some of the students will usually have discovered another commonly used construction method. This method, dubbed the *Popcorn Box*, is shown in Figure 2.

When students first look at this method, they usually choose a box with a square horizontal cross section and the 12 by 12 piece of cardboard that worked for the Pizza Box. Without calculus, they discover that this produces 81 cubic inches of volume—quite an improvement over the previous maximum of 64 cubic inches.

Next, they usually try one of two methods: either they keep the 12 by 12 piece of cardboard and allow the width to vary, or they keep the square base on the box and allow the dimensions of the 144 square inch cardboard to vary. Surprisingly, both methods produce the same maximum volume, $48\sqrt{3} \approx 83.14$. Is this true in general?

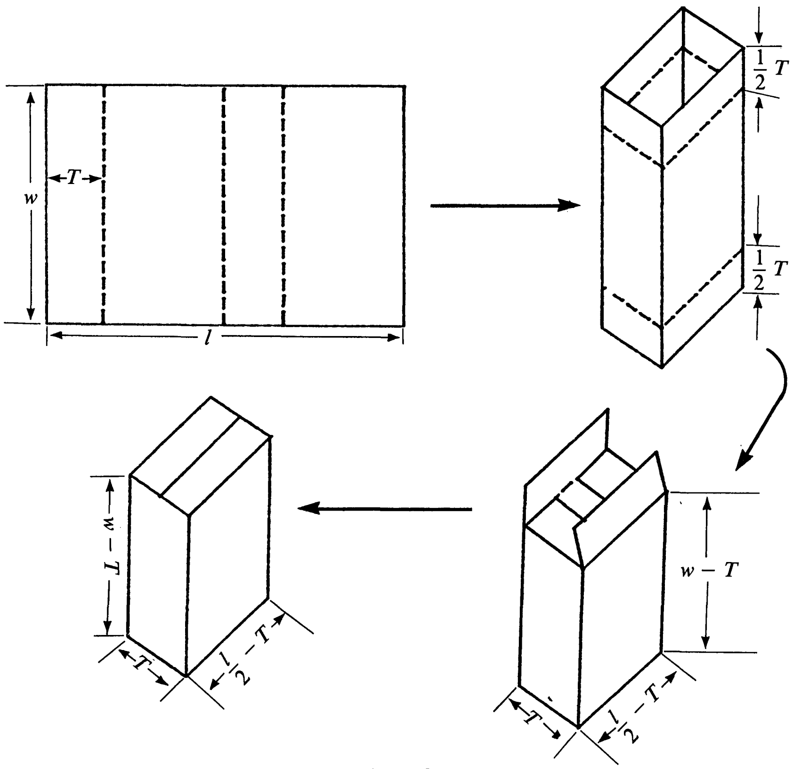



Figure 2.

To check this out for the general case, assume that the cardboard has area $A$ square inches. For $l = w = \sqrt{A}$, we have

$$V(T) = T\left(\sqrt{A} - T\right)\left(\tfrac{1}{2}\sqrt{A} - T\right)$$

and the maximum volume occurs for $T = (3 - \sqrt{3})\sqrt{A}/6$. For the second alternative, let $T = l/4$ (the box has a square base) giving

$$V(l) = (l/4)^2\left(\tfrac{A}{l} - \tfrac{l}{4}\right)$$

and maximum volume when $l = 2\sqrt{A/3} = 4w/3$. In both cases, the maximum volume is $V = \sqrt{3}\,A^{3/2}/36$.

The volume of 83.14 isn't much better than the volume of 81 that was obtained without using any calculus. However, when both the shape of the original piece of cardboard and the width $T$ of the box are allowed to vary, the improvement is more dramatic.

To see this, assume again that the area of the cardboard is $A$ square inches. As with the Pizza Box, first fix the box's width at $T$ and let the length of the cardboard vary. This gives

$$V(l) = T(\tfrac{l}{2} - T)(\tfrac{A}{l} - T),$$

and $V'(l) = 0$ when $l = \sqrt{2A}$. Using this $\sqrt{2A}$ by $\sqrt{2A}/2$ cardboard (recall that the area was fixed at $A$ square inches), allow the box width $T$ to vary. Under these conditions,

$$V(T) = T\left(\frac{\sqrt{2A}}{2} - T\right)\left(\frac{A}{\sqrt{2A}} - T\right) = T\left(\frac{1}{2}\sqrt{2A} - T\right)^2$$

and $V'(T) = 0$ when $T = \sqrt{2A}/6$. Thus, the Popcorn Box has maximum volume $V_{\text{popcorn}} = \sqrt{2}\,A^{3/2}/27$. It is now clear that $V_{\text{popcorn}}$ is approximately 41% larger than $V_{\text{pizza}}$.

After spending a class period and a daily assignment on box problems, I like to include a box problem on the next unit test. Usually I give them a specific $l$ by $w$ rectangle, tell them which method of construction to use, and ask them to find the maximum volume. As a test question, I prefer integers for $l$ and $w$, and rational values for the optimal box dimensions. The following developments show how to choose $l$ and $w$ to accomplish this for the Pizza Box and then for the Popcorn Box.

From Figure 1, we have

$$V(T) = T(l/2 - T)(w - 2T) = (lwT/2) - (l + w)T^2 + 2T^3.$$

Therefore, $V'(T) = 0$ when

$$T = \left(l + w \pm \sqrt{l^2 - lw + w^2}\right)/6.$$

The correct $T$ value will be rational when $l^2 - lw + w^2$ is a perfect square. Choosing correct $l$ and $w$ values to accomplish this result is an interesting problem whose solution has been published by the author in an earlier paper "Quasi-Pythagorean Triples for an Oblique Triangle," the TYCMJ 8 (1977), 152–155. The problem is related to the "ambiguous case" triangle pictured in Figure 3.

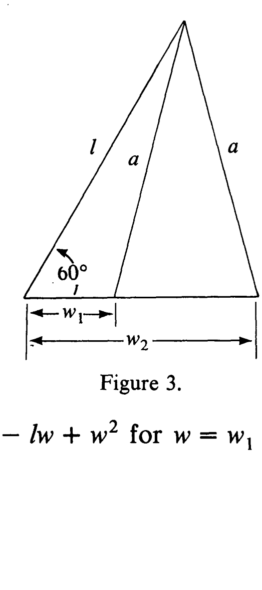



Figure 3.

The cosine law gives $a^2 = l^2 - lw + w^2$ for $w = w_1$ or $w = w_2$. Direct substitution shows that $a = m^2 + mn + n^2$ when

$$l = 2mn + m^2, \quad w_1 = m^2 - n^2 \quad \text{and} \quad w_2 = 2mn + n^2,$$

where $m > n$. It is more difficult to show that all solutions are generated by multiples of these when $m$ and $n$ are relatively prime and do not differ by a multiple of three. The net result is that for $m > n$, the pairs

$$(l, w) = (2mn + m^2, 2mn + n^2) \quad \text{and} \quad (l, w) = (2mn + n^2, m^2 - n^2)$$

generate all the Pizza Box problems one needs.

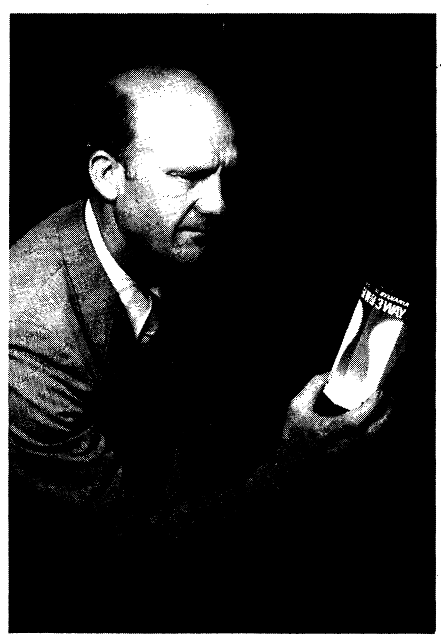
For the Popcorn Box, referring to Figure 2, we have

$$V(T) = T(l/2 - T)(w - T) = (lwT/2) - (l/2 + w)T^2 + T^3.$$

Thus, $V'(T) = 0$ when

$$T = \left(l + 2w \pm \sqrt{l^2 - 2lw + 4w^2}\right)/6.$$

In this case, $l^2 - l(2w) + (2w)^2$ needs to be a perfect square. This is the same problem as above with $w$ replaced by $2w$. It follows that $(l, 2w) = (2mn + m^2, 2mn + n^2)$ and $(l, 2w) = (2mn + m^2, m^2 - n^2)$ generate the desired dimensions.



This article would have ended here if I had not recently purchased a three way light bulb. It was packaged in an interesting box whose construction is indicated in Figure 4.

The horizontal cross-section of this box is hexagonal, and the ends are folded over just enough to reach the center. This provided a new direction to go in search of a better box.

I assigned an extra credit problem to my class to find the maximum volume using this construction method and 144 square inches of cardboard. Nobody solved the problem, but I'll try again next semester. In my solution, the volume is computed by multiplying the area of six equilateral triangles by the height. This gives the formula

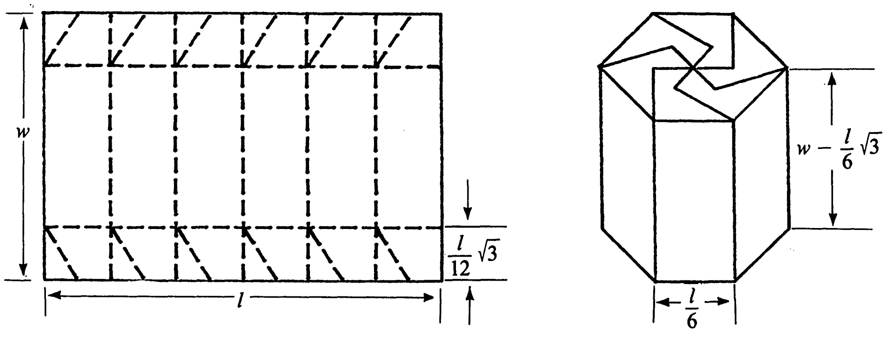




Figure 4.

$$V(l) = 6(1/2)(l/6)(l\sqrt{3}/12)(144/l - l\sqrt{3}/6).$$

Then $V'(l) = 0$ when $l = 4\sqrt[4]{108}$, and the maximum volume is $48\sqrt[4]{12} \approx 89.34$.

Many questions could be asked at this point. If we retain the $A$ square inch rectangular piece of cardboard, what is the maximum volume possible using the hexagonal cross section? If more sides are used, will the maximum volume increase? Is some number of sides optimum, or does some smooth curve eventually produce the "best" box?

These questions can be answered under the following restrictions: The polygonal cross-section must be equiangular and have $2n$ sides for some natural number $n > 1$; each of $2n - 2$ sides have length $y$ and the remaining 2 sides have length $(l/2) - (n - 1)y$, where $l$ is the length of the original rectangle. Thus, in the cross-section (see Figure 5), each of the $2n - 2$ isosceles triangles has its vertex angle equal to $\pi/n$ and its altitude of length $(y/2)\cot(\pi/2n)$.

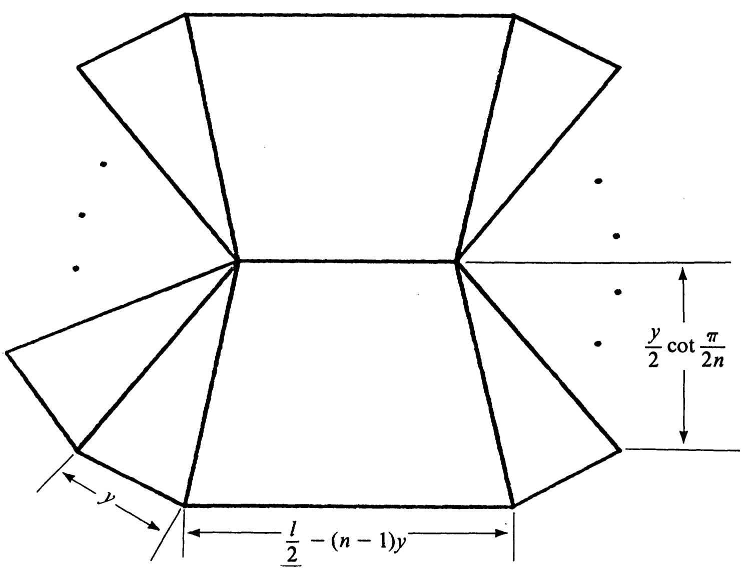




Figure 5.

For the case where $A = 144$, the box has volume

$$V = (y/2)(\cot(\pi/2n))(l - ny)(144/l - y\cot(\pi/2n)).$$

To see this, observe first that the last factor is the height of the box. The cross sectional area is seen when you place the two trapezoids and the $2n-2$ triangles side by side, with half of the triangles and one trapezoid in the inverted position. This gives a parallelogram with length $l - ny$ and altitude $(y/2)\cot(\pi/2n)$. Taking partial derivatives with respect to $l$ and $y$, we find that maximum volume

$$V = 128\sqrt{\cot(\pi/2n)}\,/\sqrt{n}$$

occurs when $l = 12\sqrt{n}\,\sqrt{\tan(\pi/2n)}$ and $y = 4\sqrt{\cot(\pi/2n)}\,/\sqrt{n}$.

The hexagonal cross section ($n = 3$) produces a maximum volume of $128\sqrt[4]{27} \approx 97.26$ cubic inches, while an octagonal cross section ($n = 4$) produces a maximum volume of $64\sqrt{\sqrt{2}+1} \approx 99.44$ cubic inches.

The preceding remarks show the maximum volume $V$ is a function of $n$. Since $V'(n) > 0$, we see that $V$ is an increasing function. Rewriting $V = 128\sqrt{\cot(\pi/2n)}\,/\sqrt{n}$ as

$$V = 128\sqrt{\frac{\pi/2n}{\sin(\pi/2n)} \cdot \frac{2\cos(\pi/2n)}{\pi}},$$

we see that $\lim_{n\to\infty} V = 128\sqrt{2/\pi} \approx 102.13$. If we could construct such a box with infinitely many sides, it would have a cross-section in the form of a rectangle with a semicircle on each end. The radius of the semicircles would be $2\sqrt{2/\pi}$, the dimensions of the rectangle would be $4\sqrt{2/\pi}$ and $2\sqrt{\pi/2}$, and the height would be $8\sqrt{2/\pi}$.

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It is important that students bring a certain ragamuffin, barefoot irreverence to their studies; they are not here to worship what is known, but to question it.

Jacob Bronowski 1908–1974
The Ascent of Man 1975 (London: BBC)