Fractions Without Quotients: Arithmetic of Repeating Decimals

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It is well known (and repeatedly taught) that a real number is rational if and only if it can be written as an infinite repeating decimal. That the decimal representation is not necessarily unique is also well known. However, if we do not allow those representations with repeating zeroes (often called terminating), the representations are unique. It is also true that there is a one-to-one correspondence between the non-zero real numbers and all non-terminating decimals (repeating or not). Strangely, zero is the only integer which cannot be written with repeating nines.

While discussing these ideas and the properties of real numbers in an intermediate algebra class, I decided to point out why we would assume the field properties of the real numbers. We had proved that multiplication and addition were closed operations on the rational numbers. Why shouldn't we at least do this much for real numbers? Since the only representations available to us of a computational nature were the infinite decimals (the number line is not very computational and Dedekind cuts were three miles over their heads), I decided to give the class some idea of the difficulties encountered in defining and performing addition and multiplication of real numbers. To this end, I asked if they could add or multiply rational numbers using only the decimal representations and their knowledge of integers.

This can be a shaking experience for some students who are very confident they have mastered "simple arithmetic" like adding and multiplying. It is well, then, to help them discover how to add repeating decimals. Also, to keep things simple, we considered only positive numbers. My class had already seen how to convert from fractions to decimals and from repeating decimals to fractions. They were aware that terminating decimals could be written with repeating nines. We used the notation of underscoring a block of digits which repeat. For example: $1/3 = .\underline{3} = .\underline{33} = .\underline{33333}$.

We looked at some examples which gave them no trouble: $.\underline{3} + .\underline{4} = 1/3 + 4/9 = 7/9 = .\underline{7}$ Then some a little more difficult:

(a) $.\underline{6} + .\underline{5} = 6/9 + 5/9 = 11/9 = 1.\underline{2}$ (not $1.\underline{1}$)
(b) $.\underline{73} + .\underline{58} = 73/99 + 58/99 = 131/99 = 1.\underline{32}$ (not $1.\underline{31}$)
(c) $1.3\underline{5} + 2.5\underline{7} = 122/90 + 232/90 = 354/90 = 3.9\underline{3}$ (not $3.9\underline{2}$)
(d) $1.0\underline{3} + 2.\underline{7} = 1.0\underline{3} + 2.7\underline{7} = 93/90 + 250/90 = 343/90 = 3.8\underline{1}$ (not $3.8\underline{0}$)

The students soon saw that if we must "carry" beyond the block of digits which repeats, we must add one to the block which then repeats in the sum. Note that we are studying a binary operation so we need only concern ourselves with adding two numbers at a time. Thus, the only number we ever carry is one! That should give you some idea how to extend this discussion to adding more than two numbers.

In the previous examples, the length of the blocks of the numbers we were adding was always the same. So, consider now: $.0\underline{54} + .\underline{98565}$

$$= .0\underline{5454545454} + .97\underline{8565985659}$$
$$= 1.0\underline{4020531114}.$$

Reason: $\underbrace{5454545454}_{\text{ten digits}} + \underbrace{8565985659}_{\text{ten digits}} = 1\,\underbrace{4020531113}_{\text{ten digits}}.$

Notice we must first get the repeating blocks to start at the same place and also have them the same length.

An obvious theorem: *If A has repeating blocks n digits in length and B has repeating blocks m digits in length then A + B has repeating blocks* L.C.M. (n, m) *digits in length.*

With the previous discussion one can easily formulate an algorithm for adding repeating decimals. Here's how it can be done:

Instructions:		Example:
(A) Write the problem.	(A)	$2.70\,\underline{584}$ $+\,6.917\,\underline{49}$
(B) Rewrite the problem so that both have repeating blocks of the same length. (See "obvious theorem.")	(B)	$2.70\,\underline{584584}$ $+\,6.917\,\underline{494949}$
(C) Rewrite problem again so that repeating blocks "start" at same position.	(C)	$2.705\,\underline{845845}$ $+\,6.917\,\underline{494949}$
(D) Add as you ordinarily would terminating decimals add 1 to the last digit if and only if you "carried" past the block which repeats.	(D–F)	9.623340794 $+\,1$ $9.623\,\underline{340795}$
(E) Underscore the last digits so that the number of underscored digits is the same as in part (B).		
(F) Simplify if possible.		

Now consider some multiplication problems;

(A) $.\underline{3} \times .\underline{4} = 3/9 \times 4/9 = 12/81 = .\underline{148}$

(B) $.\underline{1} \times .\underline{2} = 1/9 \times 2/9 = 2/81 = .\underline{024691358}$!!!

Think your students will find an algorithm for multiplication? Mine didn't. Let us find one. Using the associative law and the commutative law:

(A) $(.\underline{3})(.\underline{4}) = (3)(4)(1/9)^2 = (12)(1/9)^2 = (12)(10-1)^{-2}$

(B) $(.\underline{1})(.\underline{2}) = (1)(2)(1/9)^2 = (2)(1/9)^2 = (2)(10-1)^{-2}$

(C) $(.\underline{37})(.\underline{54}) = (37)(54)(1/99)^2 = 1998(10^2-1)^{-2}$

(D) $$\begin{aligned}(.\underline{102})(.1\underline{02}) &= (102/999)(101/990) = (102102/999999)(1020201/9999990)\\ &= (102102)(1020201)(10^6-1)^{-2}(10)^{-1}\\ &= (104164562502)(10^6-1)^{-2}(10)^{-1},\end{aligned}$$

or better yet

$$(.\underline{102})(.1\underline{02}) = (.\underline{102})(1+.\underline{02})(10)^{-1} = [.\underline{102} + (.\underline{102})(.\underline{02})](10)^{-1}$$

and $$\begin{aligned}(.\underline{102})(.\underline{02}) &= (102/999)(2/99) = (102102/999999)(020202/999999)\\ &= (102102)(20202)(10^6-1)^{-2} = 2062664604(10^6-1)^{-2}.\end{aligned}$$

Thus to do multiplication of repeating decimals we must be able to multiply integers, find integer multiples of repeating decimals, and be able to evaluate the form $(10^n - 1)^{-2}$.

Let us see how we can multiply an integer by a repeating decimal. Here are two examples:

(A) $$\begin{array}{rl} 2358 \times .\underline{67} = & (2358)(67)(.\underline{01}) \\ = & (157986)(.\underline{01}) \\ = & 01010.\underline{10} \\ + & 0505.\underline{05} \\ + & 070.\underline{70} \\ + & 09.\underline{09} \\ + & 0.\underline{80} \\ + & .\underline{06} \\ \hline = & 01595.\underline{80} \\ & +.\underline{01} \quad \text{(since 1 was carried past the block)} \\ \hline & 1595.\underline{81} \end{array}$$

(B) $$\begin{array}{rrl} \sum_{i=0}^{98} 10^{2i} \times .\underline{01} = & \underbrace{010101\ldots0101}_{198\text{ digits}} \times .\underline{01} & \\ = & 01010101\ldots010101.\underline{01} & i=98 \\ + & 010101\ldots010101.\underline{01} & i=97 \\ + & 0101\ldots010101.\underline{01} & i=96 \\ + & 01\ldots010101.\underline{01} & i=95 \\ & \vdots & \\ + & 010101.\underline{01} & i=3 \\ + & 0101.\underline{01} & i=2 \\ + & 01.\underline{01} & i=1 \\ + & .\underline{01} & i=0 \\ \hline & 01020304\ldots969798.\underline{99} = 0102030405\ldots969799. & \end{array}$$

There is a reason for doing the previous rather messy example. We need to be able to evaluate $(10^n - 1)^{-2}$. Let us do this for $n = 2$.

$$(10^2-1)^{-2} = (1/99)^2 = (.\underline{01})^2 = (.\underline{01})(1/99) = .\underline{01} \times \frac{010101\ldots01}{999999\ldots99}$$

$$= .\underline{01} \times \frac{\sum_{i=0}^{98} 10^{2i}}{\sum_{i=0}^{98} (99)10^{2i}} = \frac{\overbrace{01020304\ldots969799}^{196 \text{ digits}}}{\underbrace{9999999999\ldots999999}_{198 \text{ digits}}} = .\underline{0001020304\ldots969799}$$

In a similar manner one may show the following:

$$(10^1-1)^{-2} = (1/9)^2 = (.\underline{1})^2 = .\underline{012345679}$$

$$(10^2-1)^{-2} = (1/99)^2 = (.\underline{01})^2 = .\underline{00\ 01\ 02\ 03\ldots95\ 96\ 97\ 99}$$

$$(10^3-1)^{-2} = (1/999)^2 = (.\underline{001})^2 = .\underline{000\ 001\ 002\ldots996\ 997\ 999}$$

$$(10^4-1)^{-2} = (1/9999)^2 = (.\underline{0001})^2 = .\underline{0000\ 0001\ 0002\ldots9996\ 9997\ 9999}.$$

So, for $(10^n-1)^{-2}$, count from zero to 10^n-2 using n digits for each counting number (and zero) and then add one to the result. This is the block which repeats.

Here is a way to multiply $(10^n-1)^{-2}$ by any natural number less than 10^n. Write: $0\ 1\ 2\ 3\ 4\ 5\ldots(10^n-1)$ but use n digits for each count, supplying zeroes when necessary. To multiply $(10^n-1)^{-2}$ by m, subtract $m+1$ from 10^n then circle this number (all n digits).

(a) Write the first uncircled number (all n digits).

(b) Count the next m uncircled numbers (n digits per count), write the number on which you land.

(c) Repeat (b), returning to the beginning when necessary (in a cyclic manner).

(d) Stop when the number immediately following the circled number appears.

When you have finished writing, you have the block which repeats.

Examples:

$7 \times (10-1)^{-2}$ Write: 0 1②3 4 5 6 7 8 9 $\quad 10-(7+1)=2$

(a) 0

(b) 08

(c) 086

(c) 0864

(c) 08641

(c)(c)(c)(d) 086419753

$7 \times (10-1)^{-2} = 7 \times .\underline{012345679} = .\underline{086419753}$ ·Neat, huh?

$33 \times (10^2-1)^{-2}$ Write (or think): 00 01 02 03 · · · 96 97 98 99 Circle 66

(a) 00

(b) 00 33

(c)&(d) 00 33 67

$$33 \times (10^2-1)^{-2} = 33 \times .\underline{00010203\ldots969799} = .\underline{003367}$$

Now let's multiply $23.0\underline{12} \times .012\underline{35}$.

$$23.0\underline{12} \times .012\underline{35} = (230.\underline{12})(12.\underline{35})(10^{-4}).$$

And

$$\begin{aligned}(230.\underline{12})(12.\underline{35}) &= (230)(12) + (230)(.\underline{35}) + (12)(.\underline{12}) + (.\underline{12})(.\underline{35}) \\ &= (230)(12) + (230)(35)(.\underline{01}) + (12)(12)(.\underline{01}) + (12)(35)(.\underline{01})^2 \\ &= 2760 + (8050)(.\underline{01}) + (144)(.\underline{01}) + (420)(.\underline{01})^2.\end{aligned}$$

$$\begin{aligned}8050 \times .\underline{01} = {} & 080.\underline{80} \\ & +00.\underline{00} \\ & +\ 0.\underline{50} \\ & +\ \ .\underline{00} \\ \hline & =81.\underline{30} \\ & +\ \ .\underline{01} \qquad \text{1 was carried} \\ \hline & =81.\underline{31}\end{aligned}$$

$$\begin{aligned}144 \times .\underline{01} = {} & 01.\underline{01} \\ & 0.\underline{40} \\ & .\underline{04} \\ \hline & 1.\underline{45}\end{aligned}$$

$$\begin{aligned}420 \times (.\underline{01})^2 &= 10 \times 42 \times (.\underline{01})^2 \\ &= 10 \times .\underline{00\ 42\ 85\ 27\ \ldots\ 15\ 58} \\ &= (10 \times .\underline{0042852770125497398224670951943679216406489133761861034588307315 58})\end{aligned}$$

So

$$\begin{aligned}23.0\underline{12} \times .012\underline{35} &= (2760 + 81.\underline{31} + 1.\underline{45} + .\underline{0428527\ \ldots\ 15580})(10^{-4}) \\ &= .2842\underline{8105\ \ldots\ 56}\end{aligned}$$

The last part is not easy but it's merely an addition problem!!!!

Something which is possibly worth mentioning: the algorithms presented do not depend on the base of the numeration system. For example: The algorithm for finding $(10^n - 1)^{-2}$ can be expressed as

$$(10^n - 1)^{-2} = \frac{\left[10^{n(10^n - 2)} \sum_{i=0}^{10^n - 2} i \cdot 10^{-in}\right] + 1}{10^{n(10^n - 1)} - 1}.$$

It can be shown that the formula is true if we replace 10 with any natural number greater than one. Any readers interested in the proof of this are invited to write me. I would be happy to supply it.

I would not be so happy to supply a proof of the algorithm for multiplying a natural number by $(10^n - 1)^{-2}$. I have proved it to myself but the proof is in no way elegant. This algorithm too can be used with any base numeration system.

A problem which I find extremely challenging (if not defeating) is: Given a repeating decimal, how can we find its multiplicative inverse? You may wish to work on that.