"The Greeks displayed an insight almost as pregnant and original as their discovery of the power of reason. The universe is mathematically designed, and through mathematics man can penetrate to that design." —Morris Kline

The **Evolution** of Mathematical Certainty

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Three propositions have been identified as the foundations upon which all of Western tradition rests from the time of the Enlightenment (c. 1775 to 1825). First, all genuine questions can be answered, and if a question cannot be answered, then it is not a question. Second, all answers are knowable, and they can be discovered by means that can be learned and taught. Third, all answers must be compatible. These are the general presuppositions of the rationalist Western tradition, whether Christian or pagan, whether theist or atheist.

The following mathematical excursion demonstrates that these propositions can never be fully realized. We endeavor to trace out how the belief in the efficacy of reason in explaining everything from Nature to our present day society evolved from the time of the Greeks to our present era. The vehicle to describe this passage will be the evolution of the idea of mathematical certainty.

Greek heritage of discussion and Euclid's *Elements*

Around 560 B.C., the Ionian philosophers, including Thales, Anaximander and Anaximenes, started speculating about Nature. They initiated a search for general principles beyond observation and tried to formulate general theorems that would explain the universe. These philosophers believed people could understand the universe by reason alone. Thales, considered the father of geometry, was the earliest to formulate general mathematical laws for measuring and to prove general geometrical propositions on angles and triangles, an approach that was later followed by Euclid. However, the first major group to offer a mathematical plan of Nature was the Pythagoreans, a school led by Pythagoras (c. 585–500 B.C.). Later Aristotle (384–322 B.C.) argued that the investigations of Nature should deduce general principles from observations—the inductive phase—and then explain the observations by deducing them from the general principles—the deductive phase.

By the time of Euclid (323–285 B.C.), Greeks valued intellectual inquiry for itself, and they were interested in the nature of a logical argument regardless of the subject it would be applied to. Their goal was to rationally explain why things are the way they are. The Greeks were soon confronted with a fundamental question: can all knowledge be verified? Aristotle, for one, answered in the negative. He said there are self-evident truths that cannot be explained. Moreover, in geometry, Aristotle said a proposition is proved when it is shown to logically follow from such truths and other proven propositions. That is, he described a method for determining when an argument had been proved in an axiomatic way. Euclid knew of these intellectual developments and magnificently incorporated them into his text the *Elements*, written about 300 B.C.

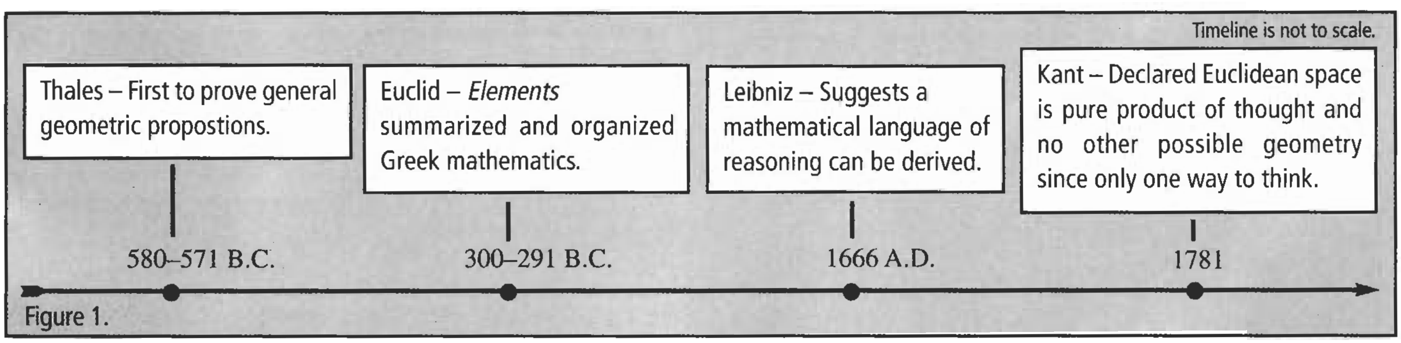

Figure 1.

From that time forward, it has been recognized as a prototype for how mathematics should be written: well-thought-out axioms, precise definitions, carefully stated theorems, and logically coherent proofs.

Formulation of the axioms or postulates is the most critical step in building an axiomatic system. From the axioms, it must be possible to deduce the interesting and important properties of the objects of study. The Greeks distinguished between general truths (axioms) and truths about geometry (postulates). In either case, the statements were to be intuitively self-evident and be acceptable without question. Through time, mathematicians began to find assumptions used in proofs in the *Elements* that were not explicitly stated in the axioms and postulates. These findings did not lessen the value of Euclid's mathematics, but merely highlighted shortcomings in the axiomatic method of the *Elements* that needed to be detailed and corrected. Credit for completely and successfully axiomatizing Euclid's geometry is given to David Hilbert.

Euclid's fifth postulate states that: If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles. This postulate may be better understood by considering one of its equivalent forms: Through a given point, not on a given line, only one parallel can be drawn to the given line. Almost immediately, this postulate, also called the *parallel postulate*, became controversial. Many did not find it to be self-evident, and because of its complexity, they thought it required a proof. Thus began a saga that was to last for over two millennia in which countless mathematicians tried to derive the parallel postulate from the others—all with no success. These futile efforts, though, began to have unexpected and important consequences in all of mathematics. See the timeline summary starting with Figure 1.

Non-Euclidean Geometry

Beginning in the eighteenth century, some mathematicians began to try indirect methods to settle the controversy surrounding the parallel postulate. Their investigations however led to new questions concerning the consistency and completeness of axiomatic systems. An axiomatic system is *consistent* if a statement and its negation cannot both be proven to be true from the axioms; a statement is *independent* of given axioms if it is impossible to either prove or disprove the statement from the other axioms; an axiomatic system is *complete* if every statement or its negation can be proven from the axioms.

In 1733, Girolamo Saccheri attempted a proof by contradiction of the parallel postulate. He did not derive a logical contradiction, but instead derived a contradiction of facts he believed to be true about the geometry of the real world. His results were a prelude to those of Carl Gauss, Nicolai Lobachevsky and John Bolyai that would follow almost one hundred years later. Lobachevsky, in 1829, showed the fifth postulate could not be proved by the first four, and furthermore that replacing it by a contrary one resulted in a consistent geometry (assuming Euclidean geometry is consistent). Bolyai also did this independently and almost simultaneously. Gauss preceded both of their efforts in developing a non-Euclidean geometry but he chose not to publish his results in part because of the poor reception he knew it would receive. Even Lobachevsky seemed to acknowledge as much when he chose to call his an "imaginary geometry." The important discovery of a non-Euclidean geometry did not receive much recognition until 1854, when G. F. B. Riemann generalized the concept of geometry and showed that various others are possible. Later on, his abstract ideas made Einstein's theory of general relativity in physics possible. Also, it was eventually seen that two-dimensional non-Euclidean geometry was simply the Euclidean geometry of some curved surfaces, spheres and pseudospheres.

With the discovery of non-Euclidean geometries, it came to be realized that mathematics could deal with completely abstract systems of axioms, which no longer had to correspond to beliefs based on real world experiences. New methods were necessary to distinguish the difference between a statement being true and being provable. The most important consideration for an axiomatic system was whether or not it was consistent. Indeed, attempts to prove that non-Euclidean geometries were invalid were essentially attempts to show that they were inconsistent. Eventually, mathematicians came to accept the validity of non-Euclidean geometries.

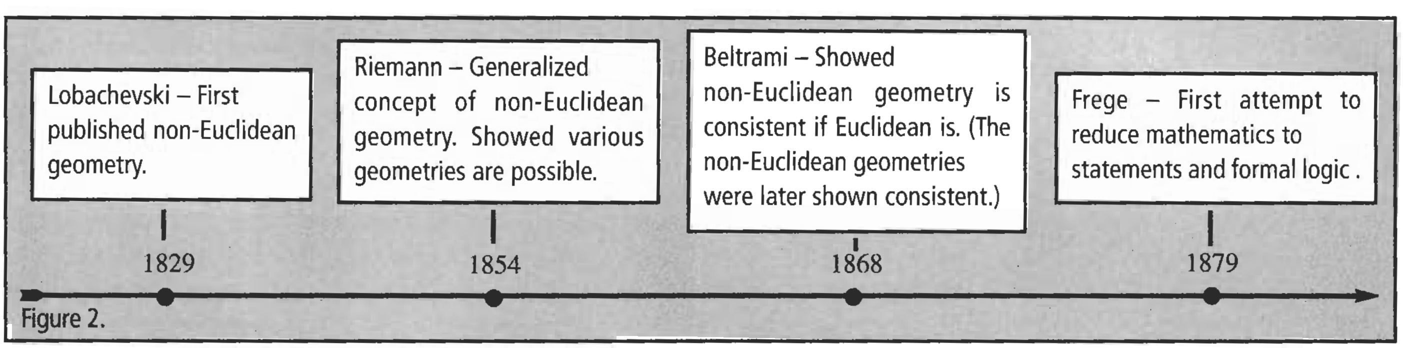

Figure 2.

At the beginning of the twentieth century, mathematicians set out to find an axiomatic system for Euclidean geometry in which the theorems of Euclid could be proved and which would stand up to the more strenuous rigor of the times. An important contributor to this goal was David Hilbert (1862-1943), one of the most important mathematicians of the century.

Hilbert's Problems and the Consistency of Mathematics

At the second International Congress of Mathematicians in 1900, Hilbert described twenty-three open problems to set the tone of mathematics for the twentieth century; they have been of central importance in mathematics since then. By this time, not only the parallel postulate, but other important assumptions were being called into question. Hilbert addressed several of these issues in his problems. (As of 2005, it is generally agreed that sixteen of the problems have been successfully settled, and that four of the remaining seven may be considered "solved" in some sense [See Benjamin Yandell's *The Honors Class: Hilbert's Problems and Their Solvers*].)

Hilbert's first problem dealt with the *continuum hypothesis*, which is a statement in set theory. When two sets have the same number of elements, they are said to have the same *cardinality*. In the 1880s, Georg Cantor (1845-1918) showed that the set of natural numbers cannot be matched up one-to-one with the set of points on a line segment, and consequently that there is more than one kind of infinity. In fact, he showed there is an endless chain of ever larger infinite sets, of ever larger cardinality. The question then arose as to whether there was a set of points with cardinality that lies between the cardinality of the natural numbers, $\aleph_0$, and the cardinality of the points on a line segment, c, this is the continuum hypothesis. This question will be dealt with shortly. Bertrand Russell soon realized that Cantor's notion of a set leads to contradictions or paradoxes in set theory. To avoid these, axiomatic set theory was developed.

There is a famous axiom in set theory, the *axiom of choice* (AC), which was used implicitly by mathematicians for years before it was explicitly described. Many did not find the AC to be self-evident and, just as the parallel postulate, it too became controversial. It states that for *any* collection of nonempty mutually exclusive sets, finite or infinite, there is a set that contains a representative member from each set. The AC with the Zermelo-Fraenkel (Z-F) axiomatic set theory is the basis of modern mathematics. In 1938, Kurt Gödel proved that if set theory without the AC is consistent, then set theory with the AC is also consistent. Moreover, Gödel proved that if set theory without the continuum hypothesis is consistent then it is also consistent with it as an axiom. And, just as it is possible to choose between different geometries in which the parallel postulate may or may not be true, it is possible to choose between different set theories in which the axiom of choice may or may not be true.

Naturally, axiomatic systems, in particular Z-F, with or without the axiom of choice or the continuum hypothesis lead to different mathematics. The decision as to which system to adopt cannot be made lightly. Theorems that require the AC are fundamental in modern analysis, topology and abstract algebra; for example, the theorem that any infinite set has a countable infinite subset. On the other hand, by adopting the AC, results such as the Banach-Tarski paradox can be derived, which says that one can divide up a golf ball into a finite number of pieces and merely by rearranging them make up a solid sphere the size of the earth!

In his second problem, Hilbert addressed the problems of consistency and completeness in mathematics. Based on the dominance of the axiomatic method in the 1800s, Hilbert felt that all mathematics should be put on a sound basis using the axiomatic method. This meant that in each field of mathematics, a set of axioms must be formulated from which the facts of the field could be proved. In 1904, he was able to construct an arithmetic model of Euclidean geometry. He and others then set out to show the consistency of arithmetic, using a finite scheme, from which it would follow that Euclidean geometry was consistent. All efforts towards achieving this goal were halted in 1931 by the unexpected results of Kurt Gödel. In his first Incompleteness Theorem, he shows that in any axiomatic system rich enough to include the arithmetic of the natural numbers, it is possible to prove some false statements implying the system is inconsistent; or it is not possible to prove some true statements implying the system is incomplete. Moreover,

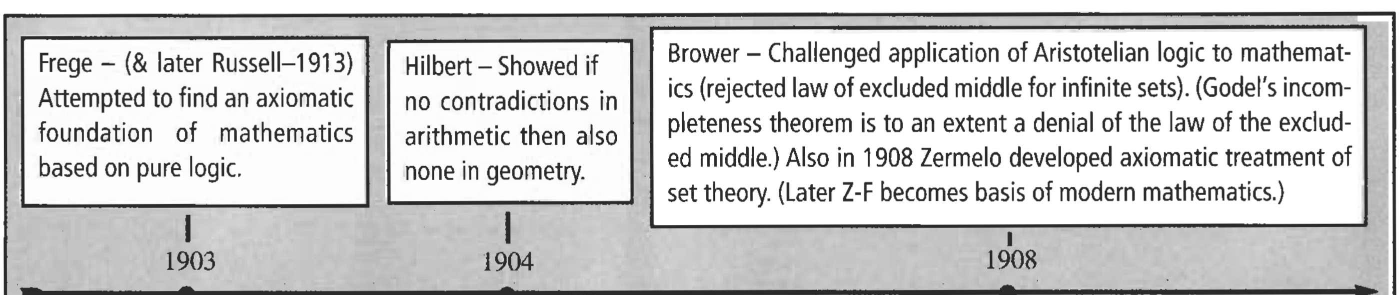

Figure 3.

in his second Incompleteness Theorem, Gödel shows that the question of whether an axiomatic system is consistent cannot be determined within the system. Thus, it is impossible to attain some of Hilbert's goals. See the associated timeline in Figures 2, 3, and 4.

More on Gödel's Incompleteness Theorems

Gödel's landmark results are important on several different levels. For one, the formal systems in which he worked are rich enough to derive all of classical mathematics. And yet within such a system, statements can be formulated that are not decidable in the system. This shortcoming remains even with the addition of more axioms to the system, no matter how many are added. Thus, Gödel showed there are limits to the axiomatic approach, in particular, with regards to verifying the consistency of arithmetic in a finite manner as proposed by Hilbert. As noted by Garrett Birkhoff, "This can mean only two things: either the reasoning by which a proof of consistency is given must contain some argument that has no formal counterpart within the system, i.e., we have not succeeded in completely formalizing the procedure of mathematical induction; or hope for a strictly '*finitistic*' proof of consistency must be given up altogether." In 1936, Gerhard Gentzen used *transfinite* induction to prove the consistency of arithmetic. It therefore remains a matter of debate as to whether a sound enough basis has been achieved for arithmetic.

Beginning in the 1920s, matters became more interesting in the field of mathematical logic when the Lowenheim-Skolem (L-S) theory was developed. As noted by Morris Kline, "Whereas Gödel's incompleteness theorem tells us that a set of axioms is not adequate to prove all the theorems belonging to the branch of mathematics that the axioms intend to cover, the L-S theorem tells us that a set of axioms permits many more essentially different interpretations than the one intended. The axioms do not limit the interpretation or models. Hence mathematical reality cannot be unambiguously incorporated into axiomatic systems." That is, if we try to uniquely capture arithmetic in a formal axiomatic system, we cannot be certain that we are describing the natural numbers or one of infinitely many different but equivalent interpretations.

As for the continuum hypothesis, it was completely resolved in 1963 by Paul Cohen with his construction of a non-Cantorian set theory. While Gödel had earlier shown that the continuum hypothesis cannot be disproved, Cohen showed that neither can it be proved. Thus it is now known that the continuum hypothesis is independent of the other Z-F axioms.

In summary, it has been shown that various foundations of mathematics can be constructed using the Z-F axioms with or without the addition of the AC or the continuum hypothesis, or their negations. Moreover, each of these is consistent if Z-F is consistent, and they result in different bodies of mathematics. The major consequence that emerges from these several conflicting approaches is that there is now not one but many mathematics. To paraphrase George Santayana, "one might say today, there is no universally accepted body of mathematics and the Greeks were its founder." See Figure 5.

Mathematical Uncertainty and Computer Science

In the proofs of his Incompleteness Theorems, Gödel exchanged problems about the provability of statements to equivalent problems about the computability of functions from the natural numbers to the natural numbers. This method required that he abstractly formalize the concept of a *computable function*. Thereafter, other mathematicians such as Alan Turing continued these investigations. In hindsight, these results can be viewed as establishing in theory the possibility that a machine could be programmed to perform various computations.

Once computers were built, many new fields of study came into existence, including *computational complexity*. Of interest here is whether a difficult problem, such as the Traveling Salesman problem, can be solved efficiently on a computer, or in more practical terms, whether such difficult problems can be solved within a reasonable time span on a computer. Similar to the successful approach utilized for the parallel postulate, the notion of an impossibility proof was utilized in the study of this problem. In 1971, Stephen Cook constructed a "highly artificial and obscure problem of propositional logic" that seemingly cannot be solved on a computer. But his problem has a most unusual property: if it could be effectively solved on a computer then so could any other equivalent problem, such as the Traveling Salesman problem. This remains an open

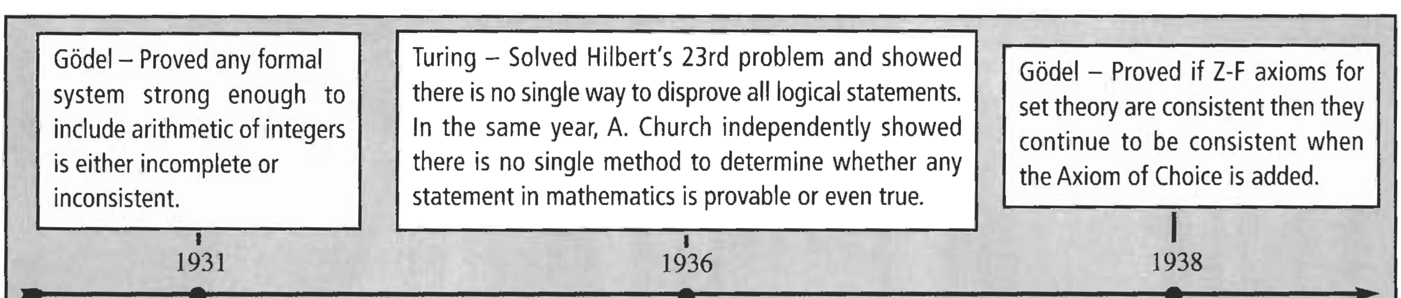

Figure 4.

question and is related to one of the Millennium Problems announced in 2000; these are meant to influence the direction of mathematics in the 21st century as Hilbert's did in the 20th century. A recap of these ideas is given in the timeline in Figure 5.

Conclusion

The Greeks were the first to attempt a rational explanation of Nature and the nature of the universe. The crucial tool in their investigations was mathematical reasoning. They assumed, and these assumptions are accepted by many even to this day, that (1) all questions about Nature and the universe can be answered by reason; (2) all answers are knowable and can be discovered; and (3) all answers are compatible. This mathematical excursion demonstrates that these goals can never be fully realized. The foregoing analysis of some "obvious" mathematical concepts has produced a cascade of never ending complications. As our investigations have shown, the quest to determine the validity of some long-held beliefs in mathematics can generate new mathematical concepts and ideas of more importance than the resolution of the original question. The truth as we see it today is this: "The laws of nature do not determine uniquely the one world that actually exists." –Hermann Weyl ■

For Further Reading

For a general history of mathematics, see Victor Katz's *A History of Mathematics*. There are some interesting discussions of the L-S theory in Morris Kline's *Mathematics: the Loss of Certainty*. More discussion of logic in mathematics can be found in Howard DeLong's *A Profile of Mathematical Logic*. The Traveling Salesman Problem and the more general problem of determining if $P = NP$ is noted by Keith Devlin in *The Millennium Problems*. The complete timeline was created using *The Timetables of Science* by Alexander Hellmans and Bryan Bunch, and also using Morris Kline's book.

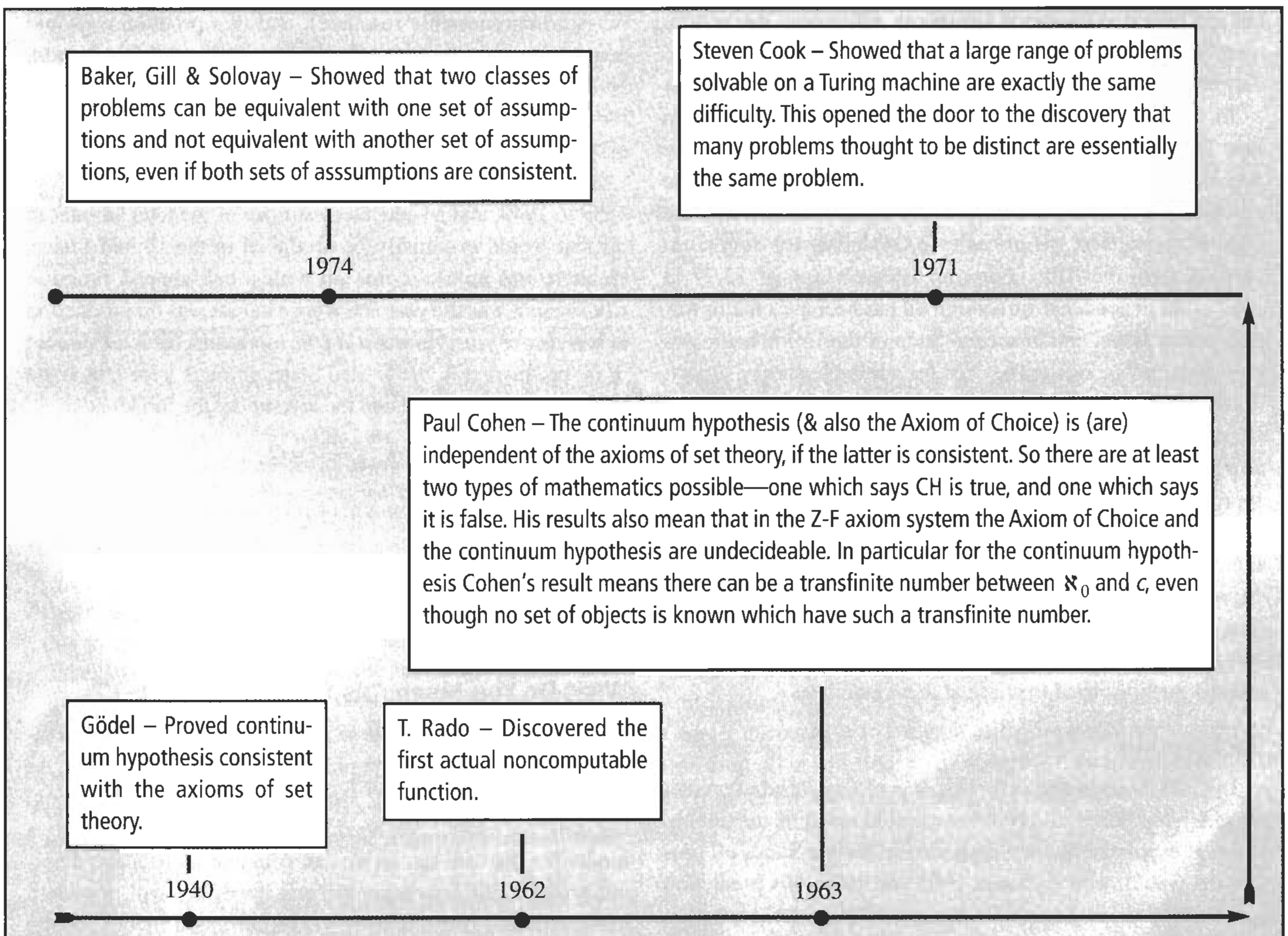

Figure 5.