IRA ROSENHOLTZ

One Point Determines a Line
A Geometric Axiom of Choice

The Axiom of Choice states that if you are given a set of non-empty sets, then it is possible to choose an element from each set. It follows that it is possible to choose a point from each and every straight line in the plane. But of course the Axiom of Choice isn't necessary for this. For example, one might choose from each vertical line its x-intercept and from each non-vertical line its y-intercept. This choice function leaves a couple of things to be desired: different lines correspond to the same point (i.e., the choice function isn't one-to-one) and some points don't correspond to any lines (i.e., the choice function is not onto). What we seek is a one-to-one and onto function from the set L of all lines in the plane to the set P of all points in the plane with the additional property that if line ℓ corresponds to point p, then p lies on ℓ. It is not difficult to believe that there is a one-to-one correspondence between the lines and the points because these sets both have cardinality c, the cardinality of the set of real numbers, although writing an explicit one down isn't that easy. But it is the additional property that makes things more fun. This article is devoted to showing that such a function exists. One nice interpretation of this is that each line uniquely determines one of its points and each point uniquely determines one of the lines through it. So that when Euclid postulated that two points determine a straight line, he wasn't being particularly efficient.

Before we begin we should mention that it isn't difficult to provide a proof that such a choice function exists based on well-orderings [1]. However, such proofs are not constructive. So, instead, we'll show that such a function exists as a consequence of Yente the Matchmaker's Marriage Theorem, which is an easy corollary of the Cantor-Schröder-Bernstein Theorem. Finally, we'll actually construct such a function—one that is very geometric in nature.

IRA ROSENHOLTZ is Professor of Mathematics at Eastern Illinois University.

All in the Family

The Cantor-Schröder-Bernstein Theorem states that if $|A| \le |B|$ and $|B| \le |A|$, then $|A| = |B|$, or, equivalently, if f is a one-to-one function from A to B and g is a one-to-one function from B to A, then there is a one-to-one and onto function h from A to B. Let's take a few moments to review one of the classical proofs of this theorem, as it will be needed later.

The first step is that we may assume that A and B are disjoint. The reason we may assume this to be the case is that we can go into the next room and use the color Xerox machine to make an aqua copy of set A and a black copy of set B. Then AquaA has the same number of elements as A and BlackB has the same number of elements as B, and AquaA and BlackB *are* disjoint. So now we are assuming that $|\text{AquaA}| \le |\text{BlackB}|$ and $|\text{BlackB}| \le |\text{AquaA}|$, and we must show that $|\text{AquaA}| = |\text{BlackB}|$. We'll simply refer to AquaA and BlackB as A and B for short, but remember that A and B now have no elements in common. The reason for doing this first step is basically so that when we take an element we will know if it is in set A or in set B without any ambiguity—it will be in one or the other, but not both. Henceforth, set A and elements of set A will be written in aqua, like A and a respectively, and set B and elements of set B will be written in bold, like **B** and **b** respectively.

Now let's continue. We are given that $|A| \le |\mathbf{B}|$, so we know that there is a one-to-one function f from A to **B**. And we are given that $|\mathbf{B}| \le |A|$, so we know that there is a one-to-one function g from **B** to A. Our job is to show that there is a *one-to-one and onto* function h from A to **B**. We will actually construct such an h, but first we'll need some preliminaries.

Now let's take an element a from set A, and look at its "family." First, since a is from set A, we may apply the function f to it to get the element **f(a)** in set **B**. (You might think of **f(a)** as being the "child" of element a.) Then since **f(a)** is in set **B**, we can apply the function g to it to get the element $g(f(a))$ back in set A. (You might think of $g(f(a))$ as being the "grandchild" of element a.) And we may continue in this fashion to produce the "descendants" of a:

Illustration by Linda L. Lawless

$$a \longrightarrow \mathbf{f(a)} \longrightarrow g(f(a)) \longrightarrow \mathbf{f(g(f(a)))} \longrightarrow \ldots$$

The descendants of a

(Note that it is possible that a can be one of its own descendants, for example if $a = g(f(a))$ or if $a = g(f(g(f(a))))$, but don't worry about that. However it is of course impossible for $a = \mathbf{f(a)}$ or $a = \mathbf{f(g(f(a)))}$.)

But this may not be all of a's family. There may be an element in set **B** which hits a under the function g. If this is the case, then there is only one element of set **B** which does this because the function g is assumed to be one-to-one. This element of set **B** (if there is one) we'll denote by $\mathbf{g^{-1}(a)}$, and we'll use a "?" to remind us that there may not be any such $\mathbf{g^{-1}(a)}$. (You might think of $\mathbf{g^{-1}(a)}$ as being the parent of a.) Similarly, the element $\mathbf{g^{-1}(a)}$ in set **B** may be hit under the function f by an element of set A which would be denoted by $f^{-1}(g^{-1}(a))$. (You might think of $f^{-1}(g^{-1}(a))$ as being the grandparent of a.) While the descendants of a certainly exist, the ancestors of a are much more iffy.

$$\ldots \mathbf{g^{-1}(f^{-1}(g^{-1}(a)))???} \longrightarrow f^{-1}(g^{-1}(a))?? \longrightarrow \mathbf{g^{-1}(a)?} \longrightarrow a$$

The ancestors of a

Or for short:

$$\ldots \mathbf{???} \longrightarrow ?? \longrightarrow \mathbf{?} \longrightarrow a$$

The ancestors of a

The descendants of a, together with the ancestors of a, together with a itself, make up a's family:

$$\ldots \mathbf{???} \longrightarrow ?? \longrightarrow \mathbf{?} \longrightarrow a \longrightarrow \mathbf{f(a)} \longrightarrow g(f(a)) \longrightarrow \ldots$$

The family of a

Elements of set A can be of three different types, depending upon what kind of families they come from. There are those elements of A which have a furthest ancestor in set A, and we denote these elements of set A by A_A. (If an element of A has no ancestors, then we'll consider that element to be in A_A.) There are those elements of A which have a furthest ancestor in set **B**, and we denote these elements of set A by A_B. And, finally, there are those elements of set A which have no furthest ancestor at all (their ancestors can be traced infinitely far back), and we denote these elements of set A by A_∞. (One is reminded of the opening line from Tolstoy's *Anna Karenina*: "All happy families are like one another; each unhappy family is unhappy in its own way.")

Similarly, each element of **B** has its own family as well, and the elements of **B** are also of three different types $\mathbf{B_A}$, $\mathbf{B_B}$ and $\mathbf{B_\infty}$, depending on whether the element of **B** has its furthest ancestor in set A, in set **B**, or has no furthest ancestor.

We are now ready to construct our long-sought-after one-to-one and onto function h from A to **B**. We define h as follows: If a belongs to A_A or A_∞, we let $\mathbf{h(a)}$ be $\mathbf{f(a)}$; if a belongs to A_B, we let $\mathbf{h(a)}$ be $\mathbf{g^{-1}(a)}$. In other words, to define $\mathbf{h(a)}$, we first look at a's family and see what kind of family it belongs to. If a has a furthest (left) ancestor which is in set A, we let $\mathbf{h(a)}$ be that member of a's family which is one place to a's right, i.e., a's child. And we do the same if a has no furthest (left) ancestor. But if a has a furthest (left) ancestor which is in set **B**, we let $\mathbf{h(a)}$ be that member of a's family which is one place to a's left, i.e., a's parent.

We must of course check that h is a one-to-one and onto function from A to **B**. It is certainly a function from A to **B**, because given an a in set A, we know exactly which element of set **B** (either $\mathbf{f(a)}$ of $\mathbf{g^{-1}(a)}$) is assigned to it. Why is it onto? Suppose **b** is an element of set **B**. Which element of set A hits it under the function h? Well, we look at **b**'s family. If **b** is in $\mathbf{B_A}$ (i.e., **b** has a furthest (left) ancestor in set A), then **b**'s parent ($f^{-1}(b)$, for those of you keeping score at home) also has it's furthest ancestor in set A, i.e., **b**'s parent is in A_A, because **b** and its parent belong to the same family. So h assigns to **b**'s parent the element of **B** immediately on its right, namely **b** itself. So this **b** is hit. The cases in which **b** is in $\mathbf{B_B}$ and **b** is in $\mathbf{B_\infty}$ are similar. Finally, the function h is one-to-one because, given a **b** in set **B**, we know exactly which element of set A hits it (depending upon **b**'s family), and there is only one element of set A which hits it because f and g are both one-to-one.

Yente's Marriage Theorem

(Yente is the matchmaker in *Fiddler on the Roof*.)

Yente the Matchmaker's Marriage Theorem. Suppose that M is a set of men (finite or infinite) and that W is a set of women (also finite or infinite). Suppose each man in M has a set of female acquaintances in W, and vice versa. Suppose that it is possible for Yente to marry off each man in M to one of his acquaintances in W so that no polygamy (polyandry) occurs. (Some of the women may end up unmarried by this pairing.) Suppose that it is also possible for Yente to marry off each woman in W to one of her acquaintances in M so that no polygamy (polygyny) occurs. (Some of the men may end up unmarried by this second pairing.) Assume further that if a is an acquaintance of b, then b is an acquaintance of a. Then it is possible for Yente to marry off *all* of the men to *all* of the women, everyone to an acquaintance, so that no polygamy of any kind occurs.

Proof. If it is possible for Yente to marry off each man in M to one of his acquaintances in W so that no polygamy (polyandry) occurs, then there is a one-to-one function f from M to W so that if $m \in M$ then $f(m)$ is one of m's acquaintances. Similarly, if it is possible for Yente to marry off each woman in W to one of her acquaintances in M so that no polygamy (polygyny) occurs, then there is a one-to-one function g from W to M so that if $w \in W$ then $g(w)$ is one of w's acquaintances. By the Cantor-Schröeder-Bernstein Theorem, there is a one-to-one and onto function from M to W, i.e., it is possible for Yente to marry off all of the men to all of the women so that no polygamy of any kind occurs. But how do we know that each person gets married to one of his or her acquaintances? The statement of the Cantor-Schröder-Bernstein Theorem doesn't tell us that. However, the PROOF does! If $m \in M$, then $h(m)$ is either $f(m)$, one of m's female acquaintances, or $g^{-1}(m)$, who has m as an acquaintance, and by assumption is therefore an acquaintance of m. Similarly, if $w \in W$, then w is married to $h^{-1}(w)$, an acquaintance of w.

A Constructive Proof

To obtain a one-to-one correspondence h from P to L so that if p is a point of the plane then p lies on the line $h(p)$, it suffices, by Yente's Theorem, to find a one-to-one function f from P to L so that if p is a point of the plane then p lies on the line $f(p)$ and a one-to-one function g from L to P so that if ℓ is a line in the plane, then the point $g(l)$ lies on ℓ. (Think of the points as being the men, the lines as being the women, and point p is an *acquaintance* of line ℓ if and only if p lies on ℓ). So our next step is finding these two one-to-one functions.

First, we define $f\colon P \to L$. If O stands for the origin, then we let $f(O)$ be the x-axis, and if p is not O, we let $f(p)$ be the line through p which is $\perp$ to Op. It is easy to see that f is one-to-one. Furthermore, it is also easy to see that the image of f consists of all lines not through the origin, together with the x-axis. (See Figure 1.)

Next we define $g\colon L \to P$. We let g (the x-axis) be O. If ℓ_1 is a line through O, which is not the x-axis, we let $g(\ell_1)$ be the point on both ℓ_1 and the open upper half of the unit circle, $U = \{(x, y) | x^2 + y^2 = 1, \text{ and } y > 0\}$. Finally, if ℓ_2 is a line not

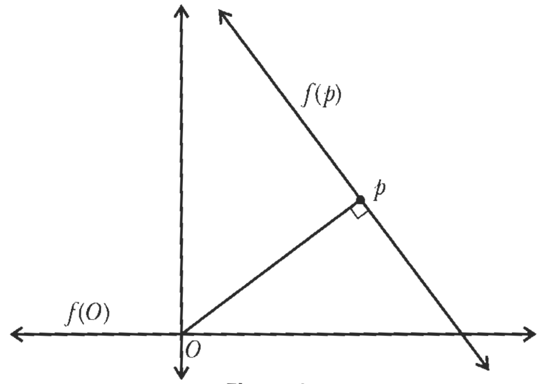

Figure 1

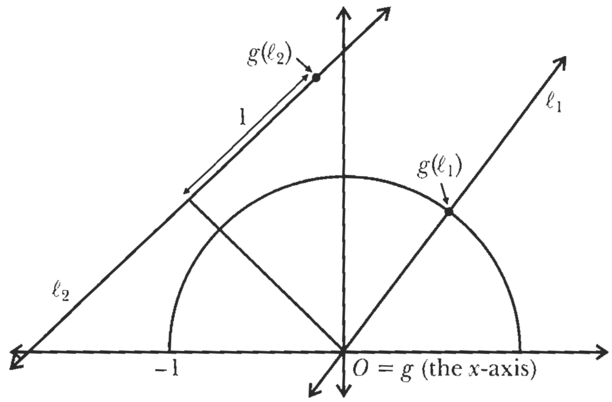

Figure 2

containing the origin, we walk from O to the point of ℓ_2 nearest the origin, make a right turn and walk to the point on ℓ_2 which is 1 unit away, and define $g(\ell_2)$ to be that point. (See Figure 2.)

Notice that, for such an ℓ_2, $g(\ell_2)$ is a point in the exterior of the unit circle—i.e., $g(\ell_2) \in E = \{(x, y) | x^2 + y^2 > 1\}$—because the hypotenuse of a right triangle is its longest side. Furthermore, any point p in the exterior of the unit circle is the image of exactly one such ℓ_2 under the function g: given p construct the circle of radius 1 about p, notice that O is

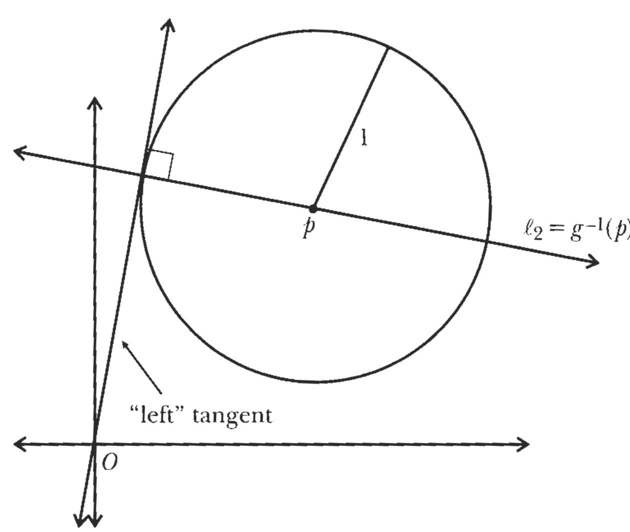

Figure 3

fine $h(p)$ to be $f(p)$. If p lies on the open dotted part of C_1, i.e., U, then p has exactly one ancestor, $p \in P_L$, so we define $h(p)$ to be $g^{-1}(p)$, the line through p and O. Finally if p lies on the part of C_n $(n > 1)$ depicted by a dotted open semicircle, then p has an odd number of ancestors, $p \in P_L$, and we define $h(p)$ to be $g^{-1}(p)$, the line through p and the point of C_{n-1} which is one unit "to the left" of p, as viewed from the origin. (The penultimate case of points on the dotted part of C_1 can be combined with the dotted case of C_n $(n > 1)$ if one considers C_0, a circle of radius 0, to be the origin.)

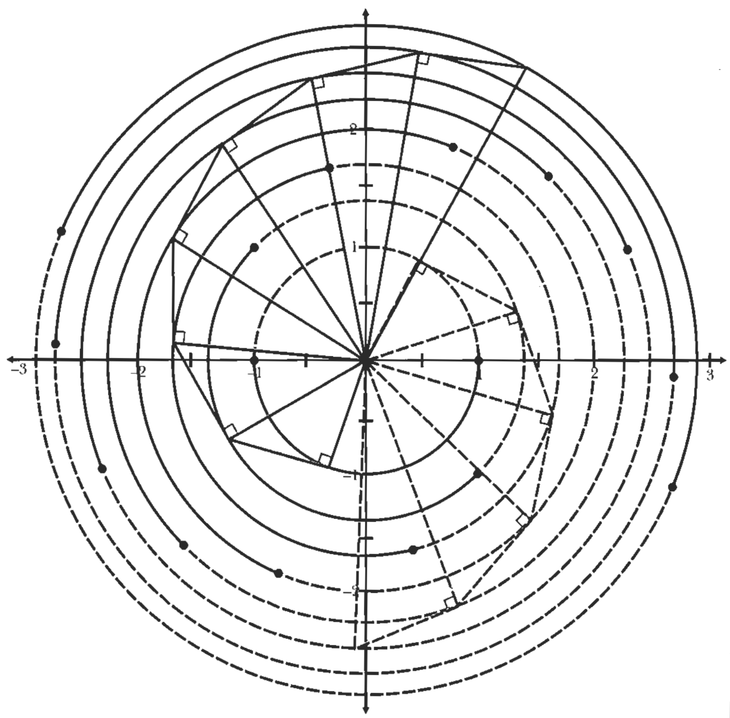

Figure 7

If you are given a point p of C_n without the entire picture, and you want to determine $h(p)$, just construct a spiral of right triangles each having a leg of length 1, spiraling to the left, until you get to the unit circle, and if you encounter a point on U, let $h(p)$ be $g^{-1}(p)$; otherwise, let $h(p)$ be $f(p)$.

This leads us to an analytic description of h, which may be preferred by some readers. The image of O under h is the x-axis. If $p = \sqrt{n}e^{i\theta}$ for some positive integer n and $\sin\left(\theta + \sum_{k=2}^{n} \arcsin\left(1/\sqrt{k}\right)\right) > 0$, then $h(p)$ is the line through p and $\sqrt{n-1}e^{i(\theta + \arcsin(1/\sqrt{n}))}$. In all other cases $h(p)$ is the line through p which is perpendicular to Op. ■

Endnote

[1] For those readers familiar with well-ordering proofs, the idea is as follows: First well-order L and P, both sets of cardinality c, so that each element of L and P has fewer than c predecessors. Next if ℓ belongs to L and assuming that from each line preceding ℓ a point has been chosen, pick the first point of ℓ not already chosen. Note that a point never gets chosen twice. Finally, to show that each point does get chosen, use the fact that a point lies on c lines, so that if it were not chosen, it must have c predecessors, a contradiction.