JOEL CHAN

As Easy As Pi

Now I, even I, would celebrate
In rhymes unapt, the great
Immortal Syracusan, rivaled nevermore,
Who in his wondrous love,
Passed on before,
Left men his guidance
How to circles mensurate.

—A. C. Orr

I leave it as an exercise for you to figure out why I included this poem about Archimedes. Perhaps I should leave a clue somewhere in this article, eh?

Archimedes, the greatest mathematician of antiquity, in his *On the Measurement of the Circle,* gave upper and lower bounds on π by inscribing and circumscribing polygons in a circle and then computing the perimeters of those polygons. The computations were quite complicated as lots of square roots were involved, so his method could not be extended indefinitely.

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Many centuries later, but still quite some time ago, while mathematicians were tediously trying to calculate π's exact value, Johann Heinrich Lambert (1728–1777) must have ticked off a few people by discovering the following theorem.

Theorem. *π is irrational.*

Poof. Consider $\tan x$, where x is a nonzero rational number.

$$x : x \in \mathbb{Q} \text{ and } x \neq 0 \Rightarrow \tan x \in \overline{\mathbb{Q}}$$

(think about it!)

$$\tan\frac{\pi}{4} = 1$$

$$\tan\frac{\pi}{4} \notin \overline{\mathbb{Q}} \Rightarrow \frac{\pi}{4} \in \overline{\mathbb{Q}}$$

$$\Rightarrow \pi \in \overline{\mathbb{Q}}.$$ QED

Whoa, you say! OK, so I had to compress the poof into four steps—the editor would have accused me of monopolizing this issue with a complete poof. Therefore I must be cruel and (sadly enough) leave it as a real hard exercise for you. (Hint: $\tan x$ can be written as a continued fraction. 'Nuff said!)

Before Lambert successfully proved π's irrationality, π had only been calculated to 112 decimal places. In 1844, Johann Martin Zacharias Dase spent two months calculating π *in his head* and correctly computed 200 decimal places. In 1987, Hideaki Tomoyori *memorized* 40,000 digits of π and recited them in only 17 hours (of course he got his name in *Guinness* for it). Today, 2,260,321,336 digits of π are known, thanks to David and Gregory Chudnovsky of Columbia University. For those who don't remember what π is, it is the ratio of a circle's circumference to its diameter, approximately

3.14159265358979323846264338327
95028841971693993751058209749
44592307816406286208998628034825
34211706798214808651328230664
70938446095505822317253594081
28481117450284102701938521105
55964462294895493038196442881097
5665933446128475648233786783l6
527120190914564856692346034861
045432664821339360726024914127
372458700660631558817488152092
096282925409171536436789259036
001133053054882046652138414695
194151160943305727036575959195
309218611738193261179310511854
807446237996274956735188575272
489122793818301194912983367336
244065664308602139494639522473
719070217986094370277053921717
629317675238467481846766940513
200056812714526356082778577134
27577896091 (No, I didn't randomly type the last 300 digits . . .)

Do the digits of π follow a random distribution? Probably, but it has not been proven yet; which is one reason why mathematicians calculate π to such great lengths. Let me now show you some of the hideous methods used to calculate π . . .

From the beginning of the 18th century, π was calculated with John Machin's formula, which can be found in almost any elementary calculus textbook:

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

Since Machin knew the power series for the arctan, the computation was done by evaluating the first bunch of terms of the power series. This formula became so powerful that most subsequent calculations of π were done by similar inverse-tangent identities. An extension of Machin's formula was used to calculate π to 1 million digits in 1973.

Since Machin's formula, other methods for calculating π were discovered. In 1914, Indian prodigy Srinivasa Ramanujan showed that

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)![1103 + 26390n]}{(n!)^4 396^{4n}}$$

(Try plugging $n = 0$ into your calculator and see what you get for π.) In 1987, Peter and Jonathan Borwein, professors at the University of Waterloo, formulated a different *Ramanujan-like convergent hypergeometric series*:

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)![212175710912\sqrt{61}+1657145277365+n(13773980892672\sqrt{61}+107578229802750)]}{(n!)^4(3n)![5280(236674+30303\sqrt{61})]^{[3n+3/2]}}$$

(Phew!) Oh, by the way, what is amazing about this formula is that each term adds about 25 digits of accuracy so by the time you reach $n = 99$, the result for π will be accurate to about 2500 digits!

With the advent of computers, mathematicians have created iterative algorithms for π such as this one, also formulated by the Borweins: Let

$$y_0 = \sqrt{2} - 1 \qquad \alpha_0 = 6 - 4\sqrt{2}$$

$$y_{n+1} = \frac{1 - (1 - y_n^4)^{1/4}}{1 + (1 - y_n^4)^{1/4}}$$

$$\alpha_{n+1} = (1 + y_{n+1})^4\,\alpha_n - 2^{2n+3}y_{n+1} \times (1 + y_{n+1} + y_{n+1}^2)\,,$$

then we have: $\dfrac{1}{\alpha_n} \to \pi$ as $n \to \infty$.

With $n = 15$, the result is guaranteed to agree with π for over 2 **billion** digits! Many recent calculations of π have used this neat algorithm.

However, the Chudnovskys used a Ramanujan-like series that they formulated eight years ago:

$$\frac{1}{\pi} = \frac{163\cdot 8\cdot 27\cdot 7\cdot 11\cdot 19\cdot 127}{640320^{3/2}}\sum_{n=0}^{\infty}\left[\frac{13591409}{163\cdot 2\cdot 9\cdot 7\cdot 11\cdot 19\cdot 127} + n\right]\frac{(6n)!}{(3n)!(n!)^3}\frac{(-1)^n}{640320^{3n}}$$

Each term only adds 14 digits of accuracy, but the beauty of the formula is that it is the fastest converging series that only uses integer terms. I will not bore you with details as to why the formula works, but it is a result from this cute observation:

$e^{\pi\sqrt{163}} =$

262 537 412 640 768 743.999999999999 2...

Well, folks, that's all for me today ... Oops! I forgot to supply a clue to the puzzle in the beginning of our chitchat. But heck, by now you should have no problem memorizing π to 30 decimal places, right? May I have a large container of coffee right now? ■

Proof vs Poof

A proof is a finite well-ordered set of statements that is supposed to convince your audience (especially students) that you know something about the given proposition.

Proofs always end with the abbreviation "QED." Originally this abbreviated the Latin phrase *Quod Erat Demonstrandum,* which means "Being what was required to be proved," or in more colloquial English "Quite Elegantly Done." All of this is what the professor thinks. Students at the University of Toronto have given the phrase their own meaning: Question Every Detail.

Poofs also originated at Toronto and the word seems to have several uses. A poof is (1) a proof that sneaks up on you and hits you like an uncountable number of bricks; then gets erased off the board before you absorb it, (2) a highly improbable construction, usually non-constructive, that produces the result by pulling a rabbit out of a hat, or (3) something which students supply, especially on exams, when asked to give a proof; such students do not usually continue in mathematics.

FURTHER READING

There is an abundance of literature dealing with π but we have space for just a few items: In the April 1992 issue of *The New Yorker* there is an article by Richard Preston about the Chudnovsky brothers, "Mountains of π," which won an award for scientific exposition from The American Association for the Advancement of Science. Dario Castellanos wrote about "The ubiquitous π" in *Mathematics Magazine,* 61(1988), 67–98 and 148–163. Finally there is a paper in *The American Mathematical Monthly* by the Borweins entitled "Ramanujan, modular equations, and approximations to pi, or how to compute one billion digits of pi" (96(1989) 201–219).