PETER SCHUMER

The Magician of Budapest

As the millennium draws to a close, many people are busy creating and debating "top ten" lists — greatest movies of all time, best fiction books of the decade, most successful musicians of the century, most significant inventions, and so on. One name which would appear on almost all such rankings of the top ten mathematicians of this century would be that of Paul Erdős (pronounced, approximately, Air Dish). In his lifetime, Erdős wrote or co-wrote nearly 1500 mathematical articles (the equivalent of a research paper every two weeks for 60 years!) He did significant work in number theory, geometry, graph theory, combinatorics, Ramsey theory, set theory, and function theory. He helped create probabilistic number theory, extremal graph theory, the probabilistic method, and much of what is now referred to as discrete mathematics. His influence on fellow mathematicians and on mathematics as a whole is bound to last for centuries to come. His reputation for being published in so many journals and in so many languages led to the following limerick:

A conjecture thought to be sound
Was that every circle was round
In a paper of Erdős
Written in Kurdish
A counterexample is found!

Epsilon Years

Paul Erdős was born in Budapest, Hungary on March 26, 1913, the son of math and physics teachers, Anna and Lajos Erdős. Paul's two older sisters died of scarlet fever while his mother remained in the hospital following his birth. This family tragedy resulted in a very close but over-protective home environment where Paul was home-schooled and his mathematical genius was able to flourish. Unfortunately, it also resulted in a socially awkward and eccentric individual who depended heavily on the care and goodwill of his many friends sprinkled across the globe.

Like the great German mathematician Carl Friedrich Gauss, Erdős's mathematical talents blossomed early. At age three Paul discovered negative numbers when he correctly subtracted 250 from 100. By age four he could multiply three-digit numbers in his head. He often entertained family friends by asking them for their birthday and then telling them how many seconds they had been alive.

At a young age Paul's father taught him two theorems about primes (i) that there are infinitely many primes, while at the same time (ii) there are arbitrarily large gaps between successive primes. To Paul, the results seemed almost paradoxical, but they led to a deep fascination with prime numbers and to a quest for a better understanding of their complicated arrangement. Again, like Gauss, an early fascination with number theory was the impetus for Erdős's lifetime dedication to the world of mathematics.

Another early mathematical influence was the *Hungarian Mathematical Journal for Secondary Schools*, commonly referred to as *KoMal*. The most popular part of the journal was a regular problem section where student solutions were pub-

Paul Erdős delivering his "Sixty Years of Mathematics" lecture at Trinity College, University of Cambridge in June, 1991, the day before receiving Cambridge's prestigious honorary doctorate. Photo by George Csicsery from his documentary film N is a Number: A Portrait of Paul Erdős.

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lished and name credit given for correct solutions. At the end of the year the pictures of the most prolific problem solvers were included in the journal. In this way, the best mathematics and science students in Hungary were introduced to one another. In this sense, Paul's "publications" began when he was barely thirteen years old.

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Paul's first significant result was a new elementary proof of Bertrand's Postulate, which he discovered as an 18-year-old university student. The French mathematician, J.L.F. Bertrand, a child prodigy himself, conjectured that for any natural number $n > 1$, there was always a prime between n and $2n$. Bertrand verified the result up to $n = 3{,}000{,}000$, but was unable to prove it. Five years later, in 1850, the Russian P.L. Chebyshev proved what has become known as Bertrand's Postulate. The proof was difficult and relied heavily on analytic methods (function theory and advanced calculus). Erdős created a new proof which, though quite intricate, was elementary in the sense that no calculus or other seemingly superfluous analytical methods were used. This proof combined with other related results on primes in various arithmetic progressions constituted his doctoral dissertation.

Another early success was a generalization of an interesting observation by one of his friends. Esther Klein noticed and proved that for any five points, no three collinear, in the plane, it is always the case that four of them can be chosen which form the vertices of a convex quadrilateral. What Erdős and George Szekeres were able to show was that for any n there is a number N, depending only on n, so that any N points in the plane (with no three collinear) have a subset of n points forming a convex n-gon. Since Esther and George became romantically involved during this period and later married, the result was dubbed the Happy End Problem. Furthermore, Erdős and Szekeres conjectured that in fact the smallest such N will always be $N = 2^{n-2}+1$. Interestingly, the more general conjecture still has not been proven. However, the Happy End Problem was a harbinger of much of Erdős's later work — fruitful collaborations and fascinating conjectures. A good theorem often creates more questions than it answers.

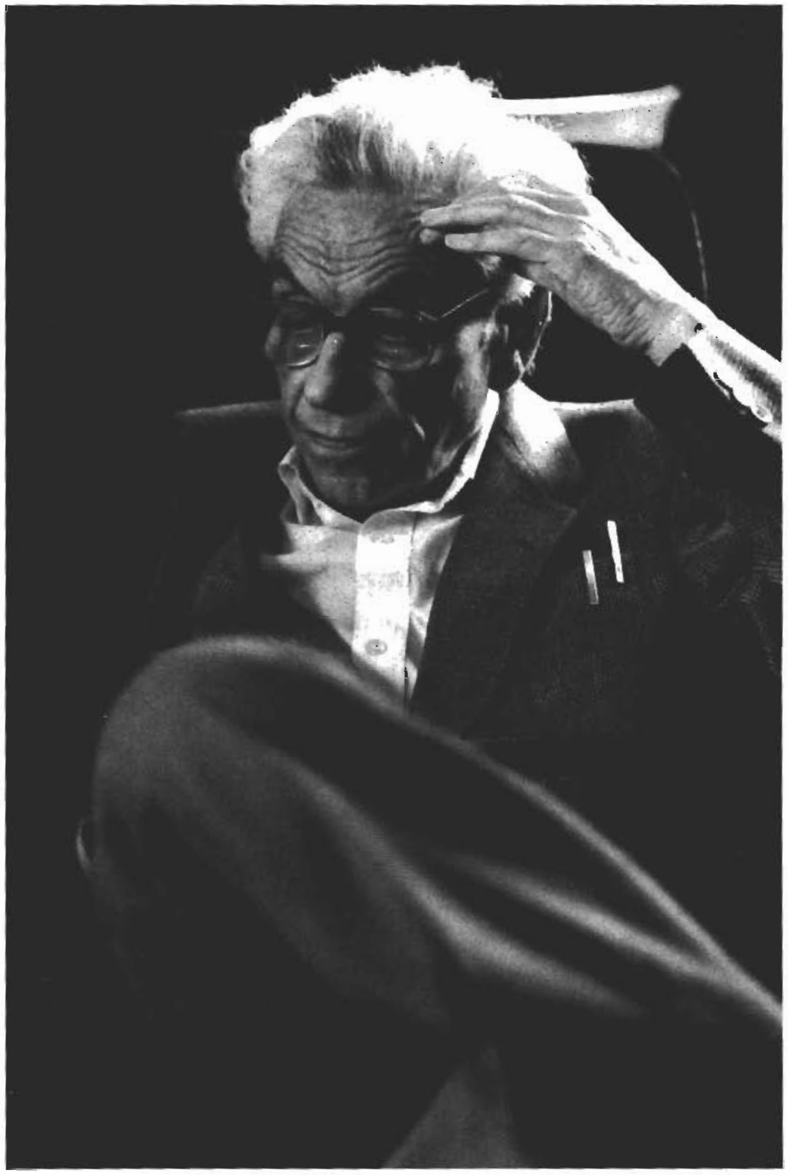


Photo by George Csicsery from his documentary film N is a Number: A Portrait of Paul Erdős.

Erdős also proved a neat result about abundant numbers. Let $s(n)$ represent the sum of the proper divisors of n (i.e., all positive divisors except n itself). Then n is called *deficient, perfect,* or *abundant* if $s(n)$ is less than, equal to, or greater than n, respectively. Such numbers have been studied since the time of Pythagoras. The German mathematician Issai Schur conjectured that the set of abundant numbers had positive density. That is, let $A(x)$ be the number of abundant numbers less than or equal to x, then Schur's conjecture states that $\lim_{x\to\infty} A(x)/x$ exists and is strictly greater than 0. Erdős's brilliant proof of this result led Schur to dub Erdős "the Magician of Budapest."

With life in Hungary ever worsening for Jewish intellectuals, Erdős obtained a fellowship to the University of Manchester in England. In 1934 Erdős traveled there via Cambridge University and began his mathematical travels and worldwide collaborations which never let up until his death. It was fourteen years before Erdős was able to return to Budapest where his mother had miraculously survived the war. Unfortunately, Paul's father died during that period and four out of five of his aunts and uncles perished in the Holocaust. Paul's beloved mother spent much of the rest of her life traveling with her son and using her apartment as a repository for his ever-increasing mountain of reprints.

Though Erdős's relationship with the American government was generally harmonious, it wasn't so during the McCarthy era. In 1954 while on a temporary faculty position at Notre Dame, Erdős wished to attend the International Congress of Mathematicians in Amsterdam. Knowing he came from a Communist country, an agent from the Immigration and Naturalization Service interviewed Erdős and asked him what he thought of Karl Marx. Erdős replied, "I'm not competent to judge. But no doubt he was a great man." Perhaps due to this, Erdős was denied a re-entry visa after attending the Mathematical Congress. Strong support and letters to state senators from many American mathematicians finally resulted in Erdős being allowed to return to the U.S. in 1959. From that point on, he could come and go freely.

His Brain is Open

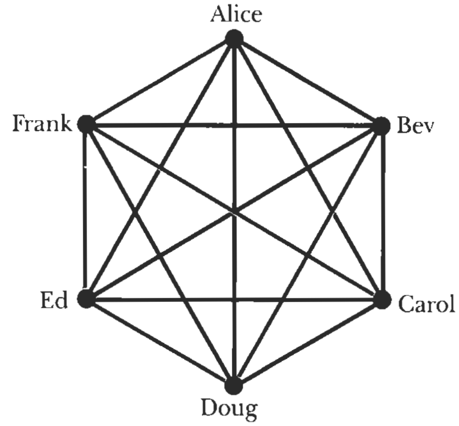
To all who knew Erdős, it appeared that he spent ninety-nine percent of his wakeful hours obsessed with mathematics (though he somehow developed a deceptive skill at both table tennis and the game of Go). Twenty hours of work a day was not at all unusual. Upon arriving at a meeting, he would announce, in his thick Hungarian accent, "my brain is open." At parties, he would often stand alone oblivious to all else, deep in thought pondering some difficult argument. When being introduced to a math graduate student, it was not unusual for him to ask, "What's your problem?" One would normally be taken aback by such a remark if it were uttered by a stranger in less than friendly surroundings, but with Erdős it was clearly meant as a friendly "hello." He was taking you seriously as a fellow dweller in the world of mathematics.

One of Erdős's greatest triumphs was his elementary proof of the Prime Number Theorem (PNT). The PNT describes the asymptotic distribution of the prime numbers and variants of it were conjectured by both Gauss and Legendre in the late 1700's. Specifically, let $\pi(x)$ be the number of primes less than or equal to x and let $Li(x) = \int_2^x dt/\log t$. The PNT states that $\lim_{x\to\infty} \pi(x)/Li(x) = 1$, that is, the number of primes less than x is asymptotic to $Li(x)$.

Significant progress towards a proof of the PNT was made by Chebyshev in the 1850's and by G.B. Riemann in 1859. Riemann's contribution was based on a deep and careful study of the complex-valued zeta function. Finally, in 1896 the French mathematician J. Hadamard and the Belgian mathematician C.J. de la Vallée Poussin each proved the result using delicate arguments from complex function theory. Many feel that the proof of the PNT was the mathematical capstone of the nineteenth century.

In the first half of the twentieth century, the search for an "elementary" proof of the PNT seemed hopeless. However, in 1949 Paul Erdős and the Norwegian mathematician Atle Selberg, working in tandem but not together, found such proofs. Selberg won a Fields Medal for his work while Erdős won the prestigious Cole Prize in algebra and number theory for his contribution.

Another area which fascinated Erdős was classical Ramsey theory which describes the number of ways of partitioning a given set into a given number of subsets. The party problem is the clas-



The party problem: if blue edges connect friends and black edges connect strangers, then Alice, Bev, and Ed are mutual strangers. There is no way to color the edges without forming either a blue or black triangle.

sic example: at a party with six individuals, prove that there must always be at least three mutual friends or three mutual strangers. More generally define $r(u,v)$ to be the smallest integer r such that if the edges of a complete graph on r vertices are colored one of two colors, then there must be a complete subgraph on u vertices of one color or a complete subgraph on v vertices of the other color. Thus, the solution to the party problem amounts to proving that $r(3,3) = 6$. The values for $r(3,n)$ are also known for $n = 4$, 5, 6, 7, and 9. In addition, it has been shown with significantly more work that $r(4,4) = 18$. Erdős showed $r(k,l) \leq C(k+l-2, k-l)$, the number of combinations of $k+l-2$ objects chosen $k-l$ at a time. Erdős offered \$250 to anyone who could prove that $\lim_{n\to\infty} r(n,n)^{1/n}$ exists. If the limit does exist, it's known to be between $\sqrt{2}$ and 4. Interestingly, the value of $r(5,5)$ is still unknown, though it must be at least 43 and at most 49. Erdős was fond of saying that if an evil spirit was going to destroy the world unless we could determine $r(5,5)$, then it would be prudent for all of humanity to devote all of its resources to this problem. On the other hand, if the evil spirit insisted on knowing $r(6,6)$, then it would be best for all of us to try to destroy the evil spirit!

Erdős not only loved working on difficult problems and proving theorems, but always strived for the most elegant and direct proof. He had unique religious views and referred to the Almighty as the

Mathematicians are often divided into two camps: theory builders and problem solvers. Erdős was definitely a problem solver. Here are a few of Erdős's many theorems.

1. There are infinitely many odd integers that are not of the form $p + 2^k$ where p is a prime.
2. The product of two or more consecutive positive integers is never a square or any other higher power.
3. A connected graph with minimum degree d and at least $2d+1$ vertices has a path of length at least $2d+1$.
4. Let p_n be the nth prime number. Then the set of limit points of the set $\{(p_{n+1} - p_n)/\log n\}$ has positive density.
5. Let $d(n)$ be the number of positive divisors of n. Then $\sum_{n=1}^{\infty} d(n)/2^n$ is irrational.

SF (or *Supreme Fascist*). Erdős felt that he was forever in the midst of an ongoing personal battle with the SF. However, one positive aspect was that the SF kept a secret book, *The Book*, which had all the theorems that would ever or could ever be discovered along with the simplest and most elegant proofs for each one. The highest compliment Erdős would give someone was that their proof was "one from The Book."

Nothing was more exciting to Erdős than to discover a mathematically talented child and to excite him or her about mathematics. In 1959 Erdős arranged to have lunch with a very precocious 11-year old, Lajos Posá. Erdős challenged the youngster to show why if $n+1$ integers are chosen from the set $\{1, 2, ..., 2n\}$, then there must be two chosen numbers which are relatively prime. Clearly the set of even numbers less than or equal to $2n$ does not have this property and shows that just choosing n such numbers is not sufficient. According to Erdős, within half a minute Posá solved the problem by making the striking observation that two consecutive integers must always be chosen. Erdős commented that rather than eating soup, perhaps champagne would have been more appropriate for this occasion!

Erdős's greatest influence on fellow mathematicians, young and old alike, was his continual outpouring of new conjectures coupled with various monetary rewards. Some problems had a price tag of just a few dollars while others went for several thousand dollars. To solve an Erdős problem is considered a great accomplishment, and the larger the reward the more difficult Erdős considered the problem to be. Erdős was often asked what would happen if all his problems were solved simultaneously. Could he possibly pay up? His answer was, "Of course not. But what would happen if all the depositors went to every bank and demanded all their money. Of course the banks could not pay — and besides, it's more likely that everyone will simultaneously ask for their money than that all of my problems will be solved suddenly."

Here are a few outstanding conjectures and open problems that Erdős left us:

1. Do there exist infinitely many primes p such that every even number less than or equal to $p-3$ can be expressed as the difference between two primes each at most p? For example, 13 is such a prime since $10 = 13-3$, $8 = 11-3$, $6 = 11-5$, $4 = 7-3$, and $2 = 5-3$. The smallest prime not satisfying this condition is $p = 97$.
2. For every integer n are there n distinct integers for which the sum of any pair is a square? For example, for $n = 5$, the numbers: -4878, 4978, 6903, 12978, and 31122 have this property.
3. Can you find a polynomial $P(x)$ for which all sums $P(a) + P(b)$ are distinct with $0 \leq a < b$? For example, $P(x) = x^3$ doesn't work since $10^3 + 9^3 = 12^3 + 1^3$. However, $P(x) = x^5$ is considered a likely candidate.
4. Are there infinitely many primes p for which $p - n!$ is composite for all n such that $1 \leq n! < p$? For example, when $p = 101$, $101 - n!$ is composite for $n = 1, 2, 3$, and 4.
5. A natural number n is *pseudoperfect* if it is abundant and it is also the sum of some of its proper divisors. For example, $n = 66$ is pseudoperfect since $66 = 11 + 22 + 33$. A number is *weird* if it is abundant but not pseudoperfect. Try verifying that $n = 70$ is a weird number (sorry Mark McGwire). For ten dollars, are there any odd weird numbers?
6. A system of congruences $a_i \pmod{n_i}$ where $n_1 < n_2 < \cdots < n_k$ is a *covering system* if every integer satisfies at least one of the congruences. For example, 0 (mod 2), 0 (mod 3), 1 (mod 4), 5 (mod 6), and 7 (mod 12) form a covering system. Erdős conjectured that for every c, there is a covering system with $n_1 \geq c$. (Currently $c = 24$ is the largest value that has been constructed.) An open question of Erdős and J. Selfridge asks whether there is a covering system with all moduli odd.
7. A \$5000 conjecture: Let $A = \{a_i\}$ be any sequence of natural numbers for which $\sum_{i=1}^{\infty} 1/a_i$ diverges. Is it true that A must contain arbitrarily long arithmetic progressions? If so, one consequence would be that the set of primes contained arbitrarily long arithmetic progressions. Currently the record is 22 consecutive primes.

Poor Great Old Man

Erdős was almost as well-known for his eccentricities as he was for his brilliant mind. By his own admission, Erdős never attempted to butter his own toast until he was already an adult. "It turned out not to be too difficult," he admitted. Many mathematicians have had the experience of either walking or driving Erdős to his next commitment, only to learn after some time that he assumed you knew where he was supposed to be going.

Erdős had an aversion to old age and infirmities and an obsession with death. When breaking off work for the night he would say, "We'll continue tomorrow — if I live." At age 60, Erdős started appending acronyms to his name. The letters *pgom* stood for "poor great old man." Five years later he added *ld* for "living dead" and so on. Eventually he got to *cd* denoting "counts dead." Erdős explained this as follows: The Hungarian Academy of Sciences has a strict limit on the total number of members that it can have at one time. However, once you reach the age of 75, while you are still a member, you are not counted against the total. Therefore, at that point, you're counted as if you were dead.

When asked "how old are you?" Erdős would answer that he was two-and-a-half billion years old. After all, when he was a child he was taught that the earth was two billion years old and now they say it's four-and-a-half billion years old! Some say that Erdős was the Bob Hope of mathematicians. Not only did he share humorous stories, but he spent a great deal of time traveling and, by lecturing, which he called *preaching*, raising the morale of the mathematical troops.

Erdős was well-known for a shorthand language many call Erdősese. An *epsilon* was a child, *poison* meant alcoholic drink (which he carefully avoided), *noise* meant music, *boss* was wife, and *slave* husband. If someone were *captured* that meant they got married, while *liberated* stood for divorced. If a mathematician stopped publishing he *died*, while actually dying was referred to as *having left*. Nothing bothered Erdős more than political strictures which did not allow for complete freedom of expression and the ability to travel freely. Erdős referred to the Soviet Union as *Joe* (for Joseph Stalin) and the United States as *Sam*.

No account of Paul Erdős would be complete without mentioning the concept of Erdős numbers. Erdős himself had Erdős number zero. Anyone who co-authored a paper with him (there are at least 472 such people) have Erdős number one. Those who do not have Erdős number one, but co-authored a paper with someone who does, are assigned Erdős number two (there are at least 5000 such people), and so on. The largest Erdős number believed to exist is seven. Erdős himself added an interesting wrinkle for those having Erdős number one. He claimed that one should instead be assigned Erdős number $1/n$ if you co-authored n papers with him. The lowest Erdős number in this case would be held by Andras Sarkozy with number 1/57 — just edging out Andras Hajnal with number 1/54. (For more on Erdős numbers, see [3].)

Having left

Paul Erdős received countless honorary degrees and his work was and continues to be the focus of many international conferences. In 1984, Erdős received the prestigious Wolf Prize for his lifetime's contributions to the world of mathematics. Of the $50,000 awarded, he immediately donated $49,280 to an Israeli scholarship named in his mother's honor. On other occasions, he donated money to Srinivasa Ramanujan's widow, to a student who needed money to attend graduate school, to a classical music station, and to several Native American causes. Always traveling with a single shabby suitcase which doubled as a briefcase, he had little need or interest in the material world. He had no home and nearly no possessions. Without hesitation, he once asked the versatile Canadian mathematician Richard Guy for a $100 adding, "You're a rich man." Richard Guy gladly gave him the money. Later Guy poignantly noted, "Yes, I was. I knew Paul Erdős."

Paul Erdős was a very versatile and creative mathematician. The vast quantity of his research output alone qualifies him as the Euler of our age. He far surpassed Einstein's ultimate litmus test for success which was to be highly esteemed by one's colleagues. Once during a lecture by the late number theorist Daniel Shanks, a long computer-generated computation resulted in a 16-digit number. Shanks, who was not known for sprinkling praise lightly said, "I don't know if anyone really understands numbers like these — well, maybe Erdős."

Paul Erdős's fantasy was that he would die in the midst of a lecture. After proving an interesting result, a voice from the audience would pipe up, "but what about the general case?" Erdős would reply, "I leave that to the next generation" and then immediately drop dead. In fact, Erdős left us on September 20, 1996 while attending a conference in Warsaw. Let's hope he is now realizing his dream of collaborating with Euclid and Archimedes simultaneously. They'd probably relish having an Erdős number of one.

Was he one of the great mathematicians of the century? Echoing Erdős himself, I'm not competent to judge. But no doubt he was a great man. ■

For More Information

1. George Csicsery, Director, *To Prove and Conjecture: Excerpts from Three Lectures by Paul Erdős*, Film, MAA.
2. ———, Director, *N is a Number: A Portrait of Paul Erdős*, Documentary Film, A. K. Peters Ltd, 1993.
3. John M. Harris and Michael J. Mossinghoff, The Eccentricities of Actors, *Math Horizons*, February 1998, 23–25.
4. Paul Hoffman, *The Man Who Loved Only Numbers*, Hyperion, 1998.
5. Bruce Schechter, *My Brain Is Open: The Mathematical Journey of Paul Erdős*, Simon and Schuster, 1998.