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A Perfectly Odd Encounter in a Reno Cafe

My father and I were sitting in a cafe in Reno. I was giving him some examples from number theory, including a problem that has been unsolved since the days of ancient Greece. Unintimidated, my father came up with a solution in about a minute flat. The reasoning behind his solution, and my skeptical reaction, reveal something about a central concern of mathematics: the nature of proof.

We were killing time while others in our party were playing the slot machines. Ted, that's my father, had been talking about the mathematics in a book he was reading, Michener's *The Source.* In fact, it wasn't really mathematics at all. It was numerology, the mystical interpretation of numerical relationships for purposes of divination. It occurred to me that number theory is the closest thing in real mathematics to the numerology Ted had been talking about, and I tried to describe the subject.

As an example, I told him about perfect numbers. The number 6 is perfect, because if you add up its proper divisors, 1, 2, and 3, the total is 6. Another example is 28: the proper divisors are 1,2,4, 7, and 14, and these sum to 28. Are there any others? Can you find some?

It has been known for over 200 years how to find all the even perfect numbers. There is a formula: $2^{n-1}(2^n-1)$. For $n = 2$ this gives $2^1(2^2-1) = 2\cdot 3 = 6$. For $n = 3$ we get $2^2(2^3-1) = 4\cdot 7 = 28$. The next possibility, $n = 4$, yields 120, and that isn't a perfect number because 60, 40, and 30 are all divisors. The trouble is that for $n = 4$, $(2^n - 1)$ isn't a prime number. Euclid showed that when (2^n-1) is prime, the formula $2^{n-1}(2^n-1)$ always produces a perfect number. Thus, for $n = 5$, $2^n-1 = 31$ is prime, so we can be sure that $2^{n-1}(2^n-1) = 496$ is perfect. Some two thousand years later, Euler proved that Euclid's formula actually generates all the even perfect numbers. So today we know the complete story on even perfect numbers.

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So what? Who cares? What possible use could there be in knowing about perfect numbers? Well, number theory is like that. It is bursting with curious relationships that aren't particularly good for anything, but which have fascinated amateur and professional mathematicians for centuries. The number theorist and the numerologist share this fascination with numbers, but the number theorist doesn't try to draw mystical conclusions from the number patterns. The object is simply to understand the mysteries and to back up each insight with proof.

Of course, it isn't always easy to find proof. That is why number theory abounds with conjectures: that is, statements fitting all the known data, and seeming to be valid general laws, but for which no proof has been found. Number theorists do not despair of ever finding proofs for these conjectures. Why, Fermat's last theorem was recently proved after standing as a conjecture for 350 years. Fermat wrote in the 1640's that $x^n + y^n = z^n$ could never hold for positive integers x, y, z, and n, with $n > 2$. That is, when working with positive integers, the sum of two cubes is never a cube, the sum of two fourth powers is never a fourth power, and so on for all powers greater than 2. From Fermat's day until our own, no proof could be found for his statement. But in 1993, Andrew Wiles announced that he had discovered such a proof, and today it is generally accepted that Fermat's theorem has been established. So, number theorists continue to hold out for proofs. No matter how overwhelming the evidence, no matter how clear the insight, true understanding is not conceded until there is proof.

That is exactly the situation with odd perfect numbers. Since Euclid's time, no one has ever been able to find an odd perfect number, even though the numbers checked by computer reach

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into the millions and beyond. It seems like an inescapable conclusion that odd numbers simply cannot be perfect and perfect numbers simply cannot be odd. But no one has been able to prove it.

So there I was, explaining to Ted about number theory, how it is like numerology, and how it is unlike. To illustrate the ideas of conjecture and proof, I told him about perfect numbers commenting that one of the oldest open questions in number theory is whether there are any odd perfect numbers. I told him, as I have told you, that the even case is completely solved. I described the state of affairs for the odd case: no one can find an odd perfect number, yet no one can prove that none exist.

Then, to my astonishment, Ted announced that it was completely obvious that an odd perfect number is an impossibility. He explained his reasoning this way: An even number is divisible by 2, and when you divide it by 2, you get one of its divisors. In fact, you get its largest possible proper divisor, half of the original number. For an even perfect number, adding up the remaining divisors produces the other half of the original number. But if the original number (n) is odd, the smallest factor (other than 1) is at least 3, so the largest proper factor is at most $n/3$. In that case, in order for n to be perfect, the other divisors—all of which are even less than $n/3$—have to add up to $2/3$ of n, and that is impossible.

Is that a proof? Was the famous problem of odd perfect numbers solved in a Reno cafe? I was instantly skeptical. Surely this argument could have occurred to Gauss, or Euler, or even *me.* But even more compelling, just from its inherent structure, I instantly realized that Ted's argument was not a proof. Can you see why?

One of the foremost skills of the trained mathematician is to recognize what is a proof, and what is not. Yet it is not always easy to clearly explain what constitutes a proof. My father's argument is logical, it is insightful, it seems to explain things. And yet there is a huge hole, a gap in the reasoning. *Why* is it impossible for there to be enough small factors to total $2/3$ of the original number? Ted could give no further explanation.

On the surface, the nature of proof seems clear cut. There must be a logical reason for each conclusion. If any of the conclusions is questioned, the prover must be able to provide reasoning that justifies it. This additional reasoning, in turn, is open to challenge, and must likewise be defended. And so on, and so forth, the prover must be prepared to provide a justification for each conclusion that is questioned. But this process cannot be taken infinitely far. At some point, won't the prover be reduced to the same position as my father? At some point, a step will be reached that is so self evident that no further explanation can be advanced. The prover can only insist that the skeptic must surely agree with the conclusion, just as my father insisted that no further argument was needed for his proof. His conclusion was transparently self evident! It was obvious! This is where deciding what is a proof gets tricky. It comes down to recognizing what is obvious, and what isn't.

Well, how *does* one recognize the obvious? It reminds me of what the Supreme Court justice said about pornography. I may not know how to define it, but I know it when I see it. Ted's final assertion was definitely not obvious. His inability to explain further invalidated the proposed proof. By training and long habit of thinking, my eye is instinctively drawn to potential gaps or holes in arguments. I constantly probe the fabric of a proof. Is there a weakness here? Can I argue a point further there? It is my years of practice that qualify me to judge what is obvious. All mathematics students need to work at this same kind of skepticism. Be wary of the obvious, distrust it, be as obtuse as you possibly can. If it is overly obvious, one ought to be able to explain why. Dig deeper, push harder, however brightly lit the corner, shine an even brighter light there, until the shadows are driven out utterly. Make this your standard practice. Only then will you be qualifed to say what is obvious.

Of course, there is little satisfaction in simply denying what someone else claims is obvious—far better to demonstrate that the desired conclusion need not follow. In the case of Ted's argument, this can be accomplished by considering the example of 945. That is an odd number whose proper divisors sum to 975. Although the largest divisor is 945/3 = 315, and all of the other divisors are even smaller, there are enough of these small divisors to add up to more than 2/3 of the original number. This is just what my father's argument claimed was impossible. This example highlights the flaw in his reasoning. And since it is possible for the proper divisors of an odd number to have a sum that exceeds the number, it might also be possible for an odd number to be perfect.

At this point, you are probably asking where the 945 came from. How did I find this example? I went looking for it. I noticed that Ted's argument, if valid, would not only rule out the possibility of odd perfect numbers, but would also rule out the existence of an odd number which is *exceeded* by the sum of the proper divisors. Also, I knew a useful fact from number theory: Given the prime factorization $m = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, the sum of all the divisors, including m itself is equal to

$$(1+p_1+p_1^2+\cdots+p_1^{e_1})\times$$
$$(1+p_2+p_2^2+\cdots+p_2^{e_2})\times\cdots\times$$
$$(1+p_n+p_n^2+\cdots+p_n^{e_n})$$

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It is easy to see that this is true. Simply observe that when the product is multiplied out, the terms are all the divisors of m. Things can be simplified a little by applying the formula

$$(1+p_i+p_i^2+\cdots+p_i^{e_i}) = \frac{p_i^{e_i+1}-1}{p_i-1}$$

This gives the sum of the divisors as

$$\frac{p_1^{e_1+1}-1}{p_1-1}\,\frac{p_2^{e_2+1}-1}{p_2-1}\cdots\frac{p_n^{e_n+1}-1}{p_n-1}$$

Using this last expression, and a hand held calculator, it is nearly effortless to compute the sum of the divisors for any number. For example, if

$$m = 3^4 \cdot 5^2 \cdot 13$$

then the sum of the divisors is $(242/2)(124/4)(168/12) = 121 \cdot 31 \cdot 14$. That's a pretty nifty way to total up the divisors of 26325!

Remember the goal is to find an odd number m which is smaller than the sum of its *proper* divisors, that is, all the divisors other than m itself. Put another way, we need the sum of all the divisors of m to exceed $2m$. Well, experiment a bit. Remember to use odd numbers, and consider some m's with two or three different prime factors, some with exponents greater than 1. It does not take long to stumble on the example

$$3^3 \cdot 5 \cdot 7 = 945.$$

And what happened to my father? Did he change his mind about odd perfect numbers? I am sorry to report that he did not. By the time I found my example, a day or two had gone by and he had moved on to other things. I don't think he recalled what the main thread of his argument had been, or indeed, what the entire dispute was about. Reno, after all, has many other diversions, and one's relatives can only be expected to sit still for so much mathematics. But I hope this article has contributed to your own understanding of proofs. And the next time something appears obvious, think of my father and the cafe in Reno where, from his point of view, the non-existence of odd perfect numbers was proved. ■

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