Upper Bounds on the Sum of Principal Divisors of an Integer

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A *prime-power* is any integer of the form p^α, where p is a prime and α is a positive integer. Two prime-powers are *independent* if they are powers of different primes. The *Fundamental Theorem of Arithmetic* amounts to the assertion that every positive integer N is the product of a unique set of independent prime-powers, which we call the *principal divisors* of N. For example, 3 and 4 are the principal divisors of 12, while 2, 5 and 9 are the principal divisors of 90. The case $N = 1$ fits this description, using the convention that an empty set has product equal to 1 (and sum equal to 0).

In a recent invited lecture, Brian Alspach noted [**1**]: *Any odd integer* $N > 15$ *that is not a prime-power is greater than twice the sum of its principal divisors.* For instance, 21 is more than twice 3 plus 7, and 35 is almost three times 5 plus 7, but 15 falls just short of twice 3 plus 5. Alspach asked for a "nice" (elegant and satisfying) proof of this observation, which he used in his lecture to prove a result about cyclic decomposition of graphs.

Responding to the challenge, we prove Alspach's observation by a very elementary argument. You, the reader, must be the judge of whether our proof qualifies as "nice." We also show how the same line of reasoning leads to several stronger yet equally elegant upper bounds on sums of principal divisors. Perhaps surprisingly, our methods will *not* focus on properties of integers. Rather, we consider properties of finite sequences of positive real numbers, and use a classical elementary inequality between the product and sum of any such sequence. But first, let us put Alspach's observation in its number theoretic context.

Aliquot parts and principal divisors

The positive divisors of a positive integer have fascinated human minds for millennia. The divisors of an integer $N > 1$ that are positive but less than N are the *aliquot parts* of N. It is usual to denote their sum by $s(N)$. Classical Greek mathematicians singled out the aliquot parts of N from among the integers less than N, by noting that N can be "built" additively from multiple copies of any one of the aliquot parts. For Euclid [**4**], a *prime* number is "that which is measured by a unit alone" (Book VII, definition 11), that is, a number which has 1 as its only aliquot part, so N is prime if $s(N) = 1$. Again, a *perfect* number is "that which is equal to its own parts" (Book VII, definition 22), that is, a number that can be "built" additively from a single copy of each of its aliquot parts, so N is perfect if $s(N) = N$. Others, such as Theon, added that N is *deficient* if $s(N) < N$, and *abundant* if $s(N) > N$. The numbers 6, 10, and 12 are examples from the three classes.

Euclid knew that there are infinitely many primes (Book IX, Theorem 20) and that an even number of the form $2^{k-1}(2^k - 1)$ is perfect when $2^k - 1$ is prime (Book IX, Theorem 36). In more modern times it has been proved that every even perfect number *must* have this form, and currently 40 such perfect numbers have been found, corresponding to the known Mersenne primes **[12]**, but it is not yet known whether there are infinitely many perfect numbers. Indeed, it is not known whether any odd perfect number exists, though many constraints on the possible form of such a number have been proved. By contrast, infinitely many positive integers are deficient and infinitely many are abundant; there can be no doubt that the Greeks knew easy proofs of these facts.

Interest in such matters underlies sophisticated modern computational studies of *aliquot sequences*, the sequences $a_0, a_1, a_2, \ldots$ beginning at a chosen positive integer $a_0 = N$, with each subsequent term found by computing the sum of the aliquot parts of the current term: $a_{k+1} = s(a_k)$ for $k \geq 0$. (See **[2, 7]** as entry points to current knowledge about aliquot sequences.) The aliquot sequence of a given N behaves in one of three possible ways: (1) after a finite number of terms it arrives at 1; (2) after a finite number of terms it enters a finite cycle, which repeats forever; (3) it continues forever without repetition. Sequences that arrive at a perfect number are of type (2), as are those that arrive at either member of a pair of *amicable* numbers, namely solutions to $s(a) = b$, $s(b) = a$. Pythagoras knew that $a = 220$ and $b = 284$ are the smallest amicable pair. Members of larger cycles are called *sociable numbers,* and several examples have been found in modern times. Intriguingly, it is not yet known whether there are any sequences of type (3); currently there are just five possible candidates with $N < 1000$, the first being $N = 276$.

It can be checked that the ratio $s(N)/N$ is 2 when $N = 120$, and is 3 when $N = 30240$. Indeed, it turns out that $s(N)/N$ has no absolute upper bound, and various simple proofs are known. When we have proved the key inequality we need in this article, we shall show that it also provides an elementary proof of this fact. (A recent Note in the MAGAZINE by Ryan **[10]** concerns the denseness of the set of numbers of the form $s(N)/N$, and of the complementary set, in the positive reals.)

Like the aliquot parts of a positive integer N, the principal divisors are a rather natural subset of the divisors of N. Indeed, if N is not a prime-power, its principal divisors are a proper subset of its aliquot parts. Thus $s^*(N)$, the sum of principal divisors of N, satisfies $s^*(N) < s(N)$ whenever N is not a prime-power. In contrast to $s(N)$, it turns out in fact that $s^*(N)$ never exceeds N. We shall prove this as our first theorem. Following common practice, we write $d|N$ when d is a positive divisor of N, and $p^\alpha \| N$ when p^α is a principal divisor of N. The Fundamental Theorem of Arithmetic implies that any positive integer N can be built multipicatively from a single copy of each of its principal divisors:

$$N = \prod_{p^\alpha \| N} p^\alpha,$$

where the notational convention is that the product ranges over all principal divisors of N. If Π is replaced by Σ, we have the sum of all principal divisors of N. It is simple and instructive to prove

THEOREM 1. *Every positive integer N satisfies*

$$N = \prod_{p^\alpha \| N} p^\alpha \geq \sum_{p^\alpha \| N} p^\alpha = s^*(N), \tag{1}$$

and (1) *holds with equality just when N is a prime-power.*

Proof. When $N = 1$, the set of all principal divisors of N is empty, so by standard conventions for empty sums and products, (1) holds with strict inequality in this case. Clearly (1) holds with equality when N has exactly one principal divisor (so N is a prime-power). Next suppose N has exactly two principal divisors, say $N = p^\alpha q^\beta$. The inequality $p^\alpha q^\beta > p^\alpha + q^\beta$ is equivalent to $(p^\alpha - 1)(q^\beta - 1) > 1$, and the latter is satisfied because $2 \le p^\alpha < q^\beta$ holds without loss of generality. Now suppose for some $k \ge 2$ that (1) holds with strict inequality for every positive integer with k principal divisors. Let N be any integer with exactly $k + 1$ principal divisors, let q^β be one of them, and let $N^* := N/q^\beta$. Then N^* has exactly k principal divisors, so

$$\begin{aligned} N = q^\beta N^* = q^\beta \prod_{p^\alpha \| N^*} p^\alpha &> q^\beta \sum_{p^\alpha \| N^*} p^\alpha = \sum_{p^\alpha \| N^*} p^\alpha q^\beta \\ &> \sum_{p^\alpha \| N^*} (p^\alpha + q^\beta) = kq^\beta + \sum_{p^\alpha \| N^*} p^\alpha > q^\beta + \sum_{p^\alpha \| N^*} p^\alpha = \sum_{p^\alpha \| N} p^\alpha. \end{aligned}$$

Hence (1) again holds with strict inequality. The theorem now follows by induction on k. ∎

From the proof of Theorem 1, we see that the inequality (1) will usually be very weak when N has several principal divisors, especially if any of them is relatively large. So could it be that N is usually at least twice as large as the sum of its principal divisors? It certainly can! This is Alspach's observation, which we mentioned at the outset:

THEOREM 2. *Let N be an odd positive integer with at least two distinct prime factors. If $N > 15$, then*

$$\frac{N-1}{2} \ge \sum_{p^\alpha \| N} p^\alpha = s^*(N). \tag{2}$$

We shall now briefly recall a classic inequality for real numbers, and then use it to prove Theorem 2.

The Bernoulli-Weierstrass inequality

Let $\mathbb{R}^+ := \{x \in \mathbb{R} : x \ge 0\}$ and, for any $n \ge 1$, let $\mathbf{a} := (a_1, a_2, \dots, a_n) \in (\mathbb{R}^+)^n$ be a sequence of nonnegative real numbers. Weierstrass **[11]** reasoned:

$$\begin{aligned} (1 + a_1)(1 + a_2) &= 1 + a_1 + a_2 + a_1 a_2 \ge 1 + a_1 + a_2, \\ (1 + a_1)(1 + a_2)(1 + a_3) &\ge (1 + a_1 + a_2)(1 + a_3) \ge 1 + a_1 + a_2 + a_3, \end{aligned}$$

and so on. Modulo attention to when equality may hold, this is essentially an inductive proof of the following theorem.

THEOREM 3. (WEIERSTRASS) *If $\mathbf{a} \in (\mathbb{R}^+)^n$ and $n \ge 1$, then*

$$\prod_{i=1}^{n} (1 + a_i) \ge 1 + \sum_{i=1}^{n} a_i, \tag{3}$$

and (3) *holds with equality if and only if at most one of the numbers a_i is nonzero.*

This classical elementary inequality (3) is the key tool underlying our arguments in this article. Some authors, such as Durell and Robson **[3]**, call it Weierstrass's inequality but it is not clear whether Weierstrass was the first to establish it. Hardy, Littlewood, and Pólya [8] noted it as Theorem 58 without attribution, though they credited Jacques Bernoulli with the special case in which **a** is a constant sequence with terms greater than -1. We shall refer to (3) as the *Bernoulli-Weierstrass inequality.*

Earlier when discussing aliquot parts we remarked that the ratio $s(N)/N$ is known to have no absolute upper bound. It is of interest here to see how this can be derived from the Bernoulli-Weierstrass inequality.

THEOREM 4. *For any integer $N \geq 2$, the sum of aliquot parts $s(N)$ satisfies*

$$\frac{s(N)}{N} \geq \sum_{p|N} \frac{1}{p}, \tag{4}$$

and (4) *holds with equality if and only if N is prime.*

Proof. The sum of all positive divisors of N is

$$\begin{aligned} N + s(N) &= \prod_{p^\alpha \| N} \sum_{p^\beta | p^\alpha} p^\beta \geq \prod_{p^\alpha \| N} (p^\alpha + p^{\alpha-1}) \\ &= \prod_{p^\alpha \| N} p^\alpha \left(1 + \frac{1}{p}\right) = N \prod_{p|N} \left(1 + \frac{1}{p}\right). \end{aligned}$$

The second step holds with equality if and only if all principal divisors of N are prime, so if and only if N is squarefree. After dividing by N, the Bernoulli-Weierstrass inequality (3) now gives

$$1 + \frac{s(N)}{N} \geq \prod_{p|N} \left(1 + \frac{1}{p}\right) \geq 1 + \sum_{p|N} \frac{1}{p}$$

with equality at the second step just when N has only one prime divisor. The stated result now follows. ■

Euler proved in 1737 that the sum of reciprocals of all primes is divergent [9], so it follows immediately from (4) that $s(N)/N$ has no absolute upper bound.

Deducing Alspach's inequality

The Bernoulli-Weierstrass inequality is really about sums and products of real numbers close to 1. We want to apply it to integers, such as occur in Alspach's inequality (Theorem 2) so we need to *scale* the individual terms to get a version of the Bernoulli-Weierstrass inequality that is about sums and products of real numbers close to some positive real number b, which we shall choose subsequently.

With $\mathbf{a} \in (\mathbb{R}^+)^n$ and $n \geq 1$, and any strictly positive $b \in \mathbb{R}^+$, multiply both sides of the Bernoulli-Weierstrass inequality (3) by b^n. Then

$$\prod_{i=1}^{n} (b + a_i b) \geq b^n + b^{n-1} \sum_{i=1}^{n} a_i b.$$

Put $\mathbf{c} := (c_1, c_2, \ldots, c_n) = (a_1 b, a_2 b, \ldots, a_n b) = b\mathbf{a}$, and choose any strictly positive $d \in \mathbb{R}^+$ such that $dn \leq b^{n-1}$. With this notation, we have

$$\prod_{i=1}^{n} (b + c_i) \geq b^n + b^{n-1} \sum_{i=1}^{n} c_i \geq d \left(bn + n \sum_{i=1}^{n} c_i \right) \geq d \sum_{i=1}^{n} (b + c_i),$$

where the last step holds with strict inequality if $n \geq 2$ and $\Sigma_{i=1}^{n} c_i > 0$. The latter fails only when $\mathbf{c} = \mathbf{0}$, where $\mathbf{0} := (0, 0, \ldots, 0) \in \mathbb{R}^+$. Hence we have a scaled version of the Bernoulli-Weierstrass inequality:

PRODUCT-SUM LEMMA. *For $n \geq 1$, any strictly positive $b \in \mathbb{R}^+$, and any sequence $\mathbf{c} \in (\mathbb{R}^+)^n$, let $N := \Pi_{i=1}^{n}(b + c_i)$. Then for any $d \in \mathbb{R}^+$ satisfying $0 < d \leq b^{n-1}/n$, we have*

$$\frac{N}{d} \geq \sum_{i=1}^{n}(b + c_i), \tag{5}$$

and (5) *holds with equality if and only if $d = b^{n-1}/n$, and $\mathbf{c} = \mathbf{0}$ if $n \geq 2$.*

In (5), note that N is a positive real, not necessarily an integer. To prove Theorem 2, we want an inequality of the form (5) with $d = 2$. But we may take $d = 2$ in the Product-Sum Lemma when $n = 2$ and $b = 4$, or when $n = 3$ and $b = \sqrt{6}$, or when $n \geq 4$ and $b = 2$. Put $\mathbf{d} := (d_1, d_2, \ldots, d_n) = (b + c_1, b + c_2, \ldots, b + c_n)$. Then $\mathbf{d}$ is a sequence of real numbers (not necessarily integers) close to b, and we have

THEOREM 5. *Let $\mathbf{d} \in (\mathbb{R}^+)^n$, with $n \geq 2$.*

(a) *If $4 \leq d_1 \leq d_2$, then $\frac{1}{2}d_1 d_2 \geq d_1 + d_2$.*

(b) *If $\sqrt{6} \leq d_1 \leq d_2 \leq d_3$, then $\frac{1}{2}d_1 d_2 d_3 \geq d_1 + d_2 + d_3$.*

(c) *If $2 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ and $n \geq 4$, then*

$$\frac{1}{2}d_1 d_2 \ldots d_n \geq d_1 + d_2 + \cdots + d_n.$$

In each case, the final relation holds with equality if and only if the preceding relations all hold with equality.

Now let us require the d_i of Theorem 5 to be distinct positive integers:

COROLLARY. *Let N be a product of $n \geq 2$ distinct positive integers $d_1 < d_2 < \cdots < d_n$. Then*

$$\frac{N}{2} > \sum_{i=1}^{n} d_i \tag{6}$$

if (a) $n = 2$ *and* $d_1 \geq 4$, *or* (b) $n = 3$ *and* $d_1 \geq 3$, *or* (c) $n \geq 4$ *and* $d_1 \geq 2$.

We are now very close to having proved Alspach's inequality, Theorem 2. In fact, we are about to obtain a more comprehensive result that also admits more than half the even integers. Since N and $\Sigma_{i=1}^{n} d_i$ are integers in the Corollary to Theorem 5, the inequality (6) is equivalent to

$$\left\lfloor \frac{N-1}{2} \right\rfloor \geq \sum_{i=1}^{n} d_i.$$

In particular, if N has $n \geq 2$ distinct prime factors, we may take the d_i to be the principal divisors of N, obtaining

$$\left\lfloor \frac{N-1}{2} \right\rfloor \geq \sum_{p^\alpha \| N} p^\alpha = s^*(N)$$

whenever the conditions of the Corollary hold: since (c) certainly holds when $n \geq 4$ and the d_i are principal divisors, the only possible exceptions are when (a) fails, so $n = 2$ and $2^1 \| N$ or $3^1 \| N$, or when (b) fails, so $n = 3$ and $2^1 \| N$. Let us settle the remaining details when $n = 2$ and $n = 3$.

When $n = 2$, let $d_1 < d_2$ be the principal divisors of N. Suppose that $d_1 = 2$, then $\frac{1}{2}N - (2 + d_2) = -2 < 0$, so $\frac{1}{2}N$ cannot exceed $d_1 + d_2$. Again, if $d_1 = 3$, then $\frac{1}{2}N - (3 + d_2) = \frac{1}{2}(d_2 - 6) > 0$ provided $d_2 \geq 7$, so $d_2 = 4$ or 5 are the exceptions. Thus, we have ruled out the cases with $N = 12, 15$ or twice an odd prime-power; in particular, every $N < 20$ with $n = 2$ is ruled out.

When $n = 3$, let $d_1 = 2 < d_2 < d_3$ be the principal divisors of N. In this case, $\frac{1}{2}N - (2 + d_2 + d_3) = (d_2 - 1)(d_3 - 1) - 3 \geq 5$, since $d_2 \geq 3, d_3 \geq 5$. Thus, there are no exceptions to the desired inequality when $n = 3$.

This completes the proof of the following result, which is more comprehensive than Theorem 2, thus achieving our original Alspach objective:

THEOREM 2A. *Any integer $N \geq 20$ with at least two distinct prime factors satisfies*

$$\left\lfloor \frac{N-1}{2} \right\rfloor \geq \sum_{p^\alpha \| N} p^\alpha = s^*(N), \tag{7}$$

except when $N = 2q^\beta$, where q is an odd prime and $\beta \geq 1$.

Note that the exceptions to (7) are actually near-misses: if $N = 2q^\beta$, then

$$\frac{N+4}{2} = \sum_{p^\alpha \| N} p^\alpha = s^*(N).$$

After checking the individual cases with $N < 20$, we deduce:

THEOREM 2B. *If N is any positive integer with at least two distinct prime factors, then*

$$\left\lfloor \frac{N+4}{2} \right\rfloor \geq \sum_{p^\alpha \| N} p^\alpha = s^*(N), \tag{8}$$

and equality holds just when $N = 2q^\beta$, where q is an odd prime and $\beta \geq 1$.

In fact, we can readily establish a stronger upper bound than (7) for $s^*(N)$, an upper bound that depends on the number of distinct prime factors in N. To achieve this, note that $n^2 \leq 3^{n-1}$ for all integers $n \geq 3$, so we may take $b = 3$ and $d = n \geq 3$ in the Product-Sum Lemma. Thus we have an extension of Theorem 5:

THEOREM 5A. *Let $\mathbf{d} \in (\mathbb{R}^+)^n$, with $n \geq 3$. If $3 \leq d_1 \leq \cdots \leq d_n$, then*

$$\frac{1}{n} d_1 d_2 \ldots d_n \geq d_1 + d_2 + \cdots + d_n,$$

and equality holds if and only if $n = 3$ and $d_1 = d_2 = d_3 = 3$.

This leads us to a result that subordinates Theorems 1, 2 and 2A:

THEOREM 6. *Any positive integer N with $n \geq 1$ distinct prime factors satisfies*

$$\frac{N}{n} \geq \sum_{p^\alpha \| N} p^\alpha = s^*(N), \tag{9}$$

except when $N = 12, 15$ or $2q^\beta$, where q is an odd prime and $\beta \geq 1$. Also (9) *holds with equality just when $N = 30$ or p^α, where p is any prime and $\alpha \geq 1$.*

Proof. The result is obvious when $n = 1$, and follows from Theorem 2A when $n = 2$. Suppose $n \geq 3$. If all principal divisors of N are at least 3, taking the d_i in Theorem 5A to be these principal divisors immediately yields (9) with strict inequality. So suppose $N = 2N^*$, where N^* is a product of $n - 1$ odd principal divisors. If $N = 30$ it is evident that (9) holds with equality. Otherwise we may assume $N^* > 15$, so $N^* \geq 21$ and

$$\frac{N^*}{n-1} > \sum_{p^\alpha \| N^*} p^\alpha = s^*(N^*)$$

follows from Theorem 2A if $n = 3$, and from Theorem 5A if $n \geq 4$. Hence

$$\frac{N}{n} = \frac{2N^*}{n} = \left(1 + \frac{n-2}{n}\right) \frac{N^*}{n-1} > \left(1 + \frac{n-2}{n}\right) \sum_{p^\alpha \| N^*} p^\alpha.$$

But $s^*(N^*) \geq 10$ and $(n-2)/n \geq \frac{1}{3}$, so

$$\frac{N}{n} > \left(1 + \frac{n-2}{n}\right) \sum_{p^\alpha \| N^*} p^\alpha > 3 + \sum_{p^\alpha \| N^*} p^\alpha > \sum_{p^\alpha \| N} p^\alpha = s^*(N).$$

Thus N satisfies (9) with strict inequality, settling all remaining cases. ■

Reverse arithmetic-geometric mean inequality

The sequence $\mathbf{a} \in (\mathbb{R}^+)^n$ with $n \geq 1$ has *arithmetic mean* $A(\mathbf{a})$ and *geometric mean* $G(\mathbf{a})$ given by

$$A(\mathbf{a}) := \frac{\sum_{i=1}^n a_i}{n} \quad \text{and} \quad G(\mathbf{a}) := \left(\prod_{i=1}^n a_i\right)^{1/n}.$$

The classical inequality comparing products and sums of finite sequences of nonnegative real numbers is the *Arithmetic-Geometric Mean Inequality*:

$$A(\mathbf{a}) \geq G(\mathbf{a}), \tag{10}$$

and (10) holds with equality if and only if $\mathbf{a}$ is a constant sequence.

A constant sequence is a scalar multiple of $\mathbf{1} := (1, 1, \ldots, 1) \in (\mathbb{R}^+)^n$, so $\mathbf{a}$ is constant precisely when $\mathbf{a} = c\mathbf{1}$ for some $c \in \mathbb{R}^+$. Given any $\mathbf{a}, \mathbf{b} \in (\mathbb{R}^+)^n$, we say that $\mathbf{a}$ ***dominates*** $\mathbf{b}$ if $a_i \geq b_i$ holds for every i in the interval $1 \leq i \leq n$, and that $\mathbf{a}$ ***strictly dominates*** $\mathbf{b}$ if furthermore the strict inequality $a_i > b_i$ holds for at least one i. Thus (10) holds with strict inequality precisely when $\mathbf{a}$ strictly dominates $c\mathbf{1}$, where $c := \min\{a_i : 1 \leq i \leq n\}$.

We may regard (10) as an inequality in which a multiple of the sum $\Sigma_{i=1}^n a_i$ is at least as large as a power of the product $\Pi_{i=1}^n a_i$. To deduce Alspach's inequality we were concerned with inequalities in the reverse direction, where a multiple of the product is at least as large as the sum. This suggests the unfamiliar novelty of comparing a multiple of $G(\mathbf{a})$ with a power of $A(\mathbf{a})$. Such an inequality does result from the Product-Sum Lemma when we take $b = n, b + c_i = a_i, d = n^{n-2}, N = G(\mathbf{a})^n$ and $\Sigma_{i=1}^n (b + c_i) = nA(\mathbf{a})$. This yields:

THEOREM 7. *For $n \geq 2$, suppose the sequence $\mathbf{a} \in (\mathbb{R}^+)^n$ dominates the constant sequence $n\mathbf{1}$. Then its arithmetic mean $A(\mathbf{a})$ and geometric mean $G(\mathbf{a})$ satisfy*

$$\frac{G(\mathbf{a})}{n} \geq \left(\frac{A(\mathbf{a})}{n}\right)^{1/n} \tag{11}$$

and (11) *holds with equality precisely when $\mathbf{a} = n\mathbf{1}$.*

For instance, $\mathbf{a} = (3, \sqrt{10}, \sqrt{10})$ strictly dominates $(3, 3, 3)$, so Theorem 7 shows that $\mathbf{a}$ satisfies (11) strictly, whence $7/2 > \sqrt{10}$. Theorem 7 yields a further extension of Theorem 5, from which we deduce two corollaries:

THEOREM 5B. *Let $\mathbf{d} \in (\mathbb{R}^+)^n$ with $n \geq 2$. If $n \leq d_1 \leq \cdots \leq d_n$, then*

$$\frac{1}{n^{n-2}} d_1 d_2 \ldots d_n \geq d_1 + d_2 + \cdots + d_n,$$

and equality holds if and only if $n = d_1 = d_2 = \cdots = d_n$.

COROLLARY 1. *If N is the product of $n \geq 2$ distinct positive integers $d_1 < d_2 < \cdots < d_n$, with $d_1 \geq n$, then*

$$\frac{N}{n^{n-2}} > \sum_{i=1}^{n} d_i. \tag{12}$$

COROLLARY 2. *If the positive integer N has $n \geq 2$ principal divisors, and each is at least as large as n, then*

$$\frac{N}{n^{n-2}} > \sum_{p^\alpha \| N} p^\alpha = s^*(N). \tag{13}$$

Extending an upper bound on $s^*(N)$

Let us review our progress. We began with the objective of finding, by elementary means, an upper bound on the sum $s^*(N)$ of principal divisors of a positive integer N. Our target upper bound was N/d with $d = 2$. We began with the modest result that the bound with denominator $d = 1$ holds without exception. Subsequently we found all N for which $d = 2$ holds. Passing from constant denominator to a linear function of n, the number of distinct prime factors of N, in Theorem 6 we found all exceptions to the simple and elegant upper bound with $d = n$. Now Corollary 2 to Theorem 5B has brought to our attention the upper bound with a super-exponential denominator $d = n^{n-2}$. In Corollary 2, that upper bound is subject to quite a strong constraint on the principal divisors of N. In this final section we complete the discussion by showing that the same upper bound holds for a much wider class of principal divisors. We use the following lemma.

MONOTONICITY LEMMA. *With $n \geq 2$, suppose $\mathbf{a} \in (\mathbb{R}^+)^n$ and $c \in \mathbb{R}^+$ satisfy*

$$c \prod_{i=1}^{n} a_i \geq \sum_{i=1}^{n} a_i > 0. \tag{14}$$

If $\mathbf{d} \in (\mathbb{R}^+)^n$ dominates $\mathbf{a}$, then

$$c \prod_{i=1}^{n} d_i \geq \sum_{i=1}^{n} d_i > 0. \tag{15}$$

Moreover, if (14) *holds with strict inequality, or if* **d** *strictly dominates* **a**, *then* (15) *holds with strict inequality.*

Proof. By (14), c and every a_i are strictly positive. Since **d** dominates **a**, every d_i is strictly positive, and the second inequality in (15) follows. Let $\delta := \max\{d_i/a_i : 1 \le i \le n\}$. For some subscript k, we have $d_k = \delta a_k$ and $d_i \ge a_i$ for all $i \ne k$, so $\Pi_{i=1}^n d_i \ge \delta \Pi_{i=1}^n a_i$. Then (15) follows, since

$$c\prod_{i=1}^{n} d_i \ge c\delta \prod_{i=1}^{n} a_i \ge \delta \sum_{i=1}^{n} a_i \ge \sum_{i=1}^{n} d_i. \tag{16}$$

If (14) holds with strict inequality, the second step in (16) is a strict inequality. Also the last step in (16) holds with equality only if $\mathbf{d} = \delta\mathbf{a}$. But then $\Pi_{i=1}^n d_i = \delta^n \Pi_{i=1}^n a_i$, implying strict inequality in the first step of (16) when **d** strictly dominates **a**, for then $\delta > 1$. The lemma follows. ∎

In Corollary 2 to Theorem 5B the upper bound on $s^*(N)$ with super-exponential denominator $d = n^{n-2}$ holds if the principal divisors of N are at least n. We now show that the same upper bound holds if the principal divisors exceed $n/2$, a much milder constraint.

THEOREM 8. *If N is any positive integer with $n \ge 2$ principal divisors, and each is greater than $n/2$, then*

$$\frac{N}{n^{n-2}} \ge \sum_{p^\alpha \| N} p^\alpha = s^*(N), \tag{17}$$

and (17) *holds with equality precisely when $N = 30$.*

Proof. Suppose N has $n \ge 2$ principal divisors, each greater than $n/2$. Let $\mathbf{d} \in (\mathbb{R}^+)^n$ be the sequence of those principal divisors in increasing order. At most one principal divisor of N is even, so **d** dominates the increasing sequence $\mathbf{a}^*(n)$, which we define to comprise the smallest even integer greater than $n/2$ and the smallest $n-1$ consecutive odd integers greater than $n/2$. We claim that

$$\frac{N}{n^{n-2}} = \frac{1}{n^{n-2}}\prod_{i=1}^{n} d_i \ge \sum_{i=1}^{n} d_i = s^*(N).$$

Using the Monotonicity Lemma, this claim will follow if we can show that

$$\frac{1}{n^{n-2}}\prod_{i=1}^{n} a_i^*(n) \ge \sum_{i=1}^{n} a_i^*(n). \tag{18}$$

For brevity we use $P(n)$ and $S(n)$, respectively, to denote the left and right sides of (18). Routine evaluation of (18) for each n in the interval $2 \le n \le 12$ shows that $P(3) = S(3)$, corresponding to $N = 30$, and $P(n) > S(n)$ in all other cases. We now prove inductively that $P(n) > S(n)$ holds for every $n \ge 9$, whence the theorem follows by the Monotonicity Lemma, with $N = 30$ as the sole instance of equality. (The overlap for $9 \le n \le 12$ is needed.)

The fine structure of $\mathbf{a}^*(n)$ depends on the residue class of n modulo 4. The even integer in $\mathbf{a}^*(n+4)$ is 2 greater than the even integer in $\mathbf{a}^*(n)$, and the odd integers in $\mathbf{a}^*(n+4)$ are all but the smallest odd integer in $\mathbf{a}^*(n)$, together with the next 5 odd integers. In particular, suppose $n = 4k + 1$ for some positive integer k. Then the ratio $P(n+4)/P(n)$ is equal to

$$\frac{2k+4}{2k+2} \cdot \frac{(10k+1)(10k+3)(10k+5)(10k+7)(10k+9)}{2k+1} \cdot \frac{n^{n-2}}{(n+4)^{n+2}}$$
$$= \frac{k+2}{k+1} \cdot \frac{10k+1}{4k+1} \cdot \frac{10k+3}{4k+1} \cdot \frac{10k+7}{4k+5} \cdot \frac{10k+9}{4k+5} \cdot \frac{5}{(1+\frac{4}{n})^n}.$$

As $k \to \infty$ the first, third and sixth factors decrease monotonically, so are always greater than their limits; the other three factors increase monotonically, so if we require $k \geq 2$ they are never less than their values at $k = 2$. Hence when $n = 4k + 1$ and $k \geq 2$ we have

$$\frac{P(n+4)}{P(n)} > 1 \cdot \frac{7}{3} \cdot \frac{5}{2} \cdot \frac{27}{13} \cdot \frac{29}{13} \cdot \frac{5}{e^4} > \frac{7}{3}. \tag{19}$$

Similarly, if $n = 4k + 1$ then

$$\frac{S(n+4)}{S(n)} = \frac{12k^2 + 25k + 14}{12k^2 + k + 1}.$$

As $k \to \infty$ this ratio decreases monotonically, so if we require $k \geq 2$ it never exceeds its value at $k = 2$, and

$$\frac{S(n+4)}{S(n)} \leq \frac{112}{51} < \frac{7}{3}. \tag{20}$$

If $P(n) > S(n)$ when $n = 4k + 1$ for some $k \geq 2$, then (19) and (20) imply

$$P(n+4) > \frac{7}{3}P(n) > \frac{7}{3}S(n) > S(n+4).$$

Since $P(9) > S(9)$, induction now guarantees that (18) holds with strict inequality when $n = 4k + 1$ for all $k \geq 2$.

Similar computations for n in the other residue classes modulo 4 complete the proof. ■

Closing remarks The Monotonicity Lemma is actually strong enough to yield a number of our earlier results. In particular, once the formulations of the Product-Sum Lemma and Theorem 7 have been discovered, they can be readily proved using the Monotonicity Lemma. It is useful for proving inequalities in which a product is greater than a sum, but is of little help in the initial task of formulating the inequalities. In recent papers **[5, 6]**, we studied an inequality between two polynomials related to the sum and product of an arbitrary sequence. Our results generalize the Bernoulli-Weierstrass inequality, so could yield inequalities like those established here. However, in the spirit of Alspach's motivating request, we tried here to keep our arguments as elementary and self-contained as possible.

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Requiescat. After a long battle with cancer, William P. Galvin passed away on December 12, 2003.

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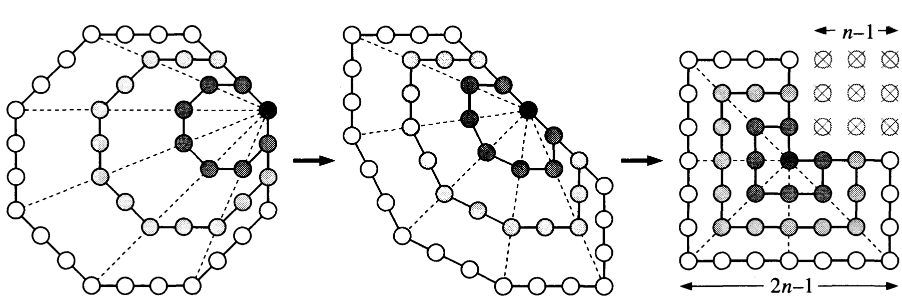
Proof Without Words: Every Octagonal Number Is the Difference of Two Squares

$$1 = 1 = 1^2 - 0^2$$
$$1 + 7 = 8 = 3^2 - 1^2$$
$$1 + 7 + 13 = 21 = 5^2 - 2^2$$
$$1 + 7 + 13 + 19 = 40 = 7^2 - 3^2$$
$$O_n = 1 + 7 + \cdots + (6n - 5) = (2n - 1)^2 - (n - 1)^2$$



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