ARTICLES

Drawing a Triangle on the Thurston Model of Hyperbolic Space

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Experiments with a common physical model of the hyperbolic plane presented the authors with surprising difficulties in drawing a large triangle. Understanding these difficulties led to an intriguing exploration of the geometry of the Thurston model of the hyperbolic plane. In this exploration we encountered topics ranging from combinatorics and Pick's Theorem to differential geometry and the Gauss-Bonnet Theorem.

The journey began when one of the authors was teaching a class of non-mathematics majors using Ed Burger and Michael Starbird's popular text *The Heart of Mathematics* [**1**]. In section 4.6, Burger and Starbird describe how to build a model of the hyperbolic plane out of paper by taping together equilateral triangles with 7 triangles around each vertex; FIGURE 1 shows the result. They then ask the following question:

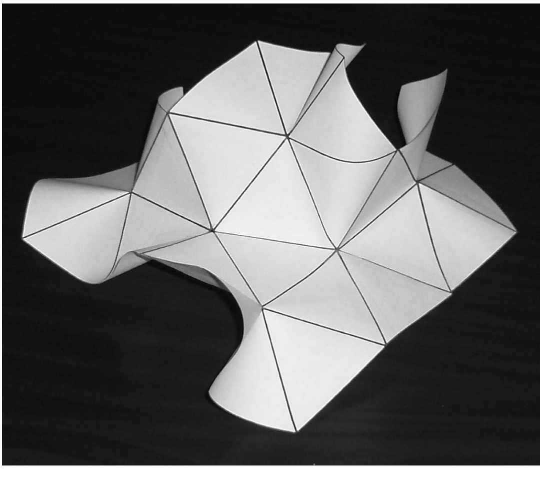

Figure 1 The Thurston model of the hyperbolic plane

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> Draw a big triangle upon your floppy sheet (the model) spanning several of the pieces by flattening a section on the ground and drawing a straight line, then flattening another section and drawing another straight line, and then completing the triangle in the same way. There is a lot of squashing involved. Now measure the three angles and add them up. What do you get? (section 4.6, problem #18)

This question is unexpectedly difficult to answer and raises interesting questions about the relationship between the model and the hyperbolic plane. For example, what is meant by a "big" triangle? And what is a "straight line"?

The model described by Burger and Starbird was initially suggested by William Thurston as a way for people to get a feeling for hyperbolic space, and has appeared in several books aimed at a general audience, in particular, *The Shape of Space* by Jeffrey Weeks [**5**, p. 151] and *The Heart of Mathematics* [**1**, p. 301]. We encourage readers to construct their own models, both to verify for themselves the results in this paper, and simply because they are very cool toys!

Notice that the Thurston model shown in FIGURE 1 cannot be flattened onto the plane because we are forcing $7\pi/3$ radians to fit around each vertex rather than the 2π radians allowed in the Euclidean plane. However, there are strips of equilateral triangles in the model that *can* be flattened onto the Euclidean plane, as shown in FIGURE 2.

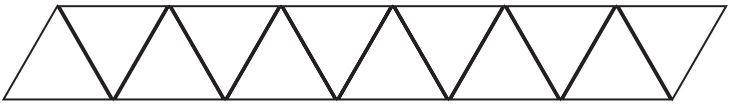

Figure 2 A strip of equilateral triangles in the Euclidean plane

When Burger and Starbird ask us to draw a "big" triangle, it is natural to think in terms of area. However, we can draw a triangle with as much area as we wish within one of these Euclidean strips of triangles, and the result will have an angle sum of π. Since the purpose of the model is to illustrate the *differences* between Euclidean and hyperbolic geometry, this is clearly not what was meant. Instead of looking at area *per se*, we want to draw a triangle containing a large number of the vertices of the model in its interior.

Before we can begin to draw any kind of triangle, big or small, we need to know what we mean by straight lines in the model. Burger and Starbird suggest we should "flatten a section [of the model] on the ground" and draw a straight line on this flattened section. But, then, what of a line that runs along the sides of one of the Euclidean strips shown in FIGURE 2? This certainly seems like a straight line—and yet, since it passes through vertices where the model *cannot* be flattened without folding the model onto itself, they cannot be drawn as Burger and Starbird describe. How should we resolve this? Answering this questions leads to some beautiful mathematics, including the Gauss-Bonnet Theorem relating the area of a hyperbolic triangle to the sum of its angles.

Drawing lines in Thurston models

Before we dive into the nitty gritty of drawing lines and triangles, we need to address to what extent the Thurston model actually models hyperbolic space. It might be better to say that it is an *approximate* model, in the same way that an icosahedron is an approximate model of the sphere. It is most natural to look at the geometry on the Thurston model induced by its embedding as a surface in $\mathbb{R}^3$; however, this geometry does not strictly satisfy the axioms of hyperbolic geometry (or even incidence geometry!). Alternatively, we can define a map from an actual hyperbolic plane to the Thurston model, and use this map to define the geometry of the model; this results in a different measure of distance, and hence in different lines and polygons. We are often interested in comparing these two perspectives. The "natural" geometry is easier to use in a classroom (as long as we place certain restrictions), so we begin from that point of view by defining *Thurston lines* (these are the lines imagined by Burger and Starbird). In subsequent sections we will define the correspondence between the Thurston model and a standard model of hyperbolic space, the *Poincaré disk model*, and use it to define a different set of lines, the *hyperbolic lines*. By comparing these two notions of lines we will see that this natural geometry, while not the same as the hyperbolic geometry, does provide a useful approximation.

Thurston lines The standard method to define a line in a space is as the shortest path (or *geodesic*) between any two points of the space. In the Thurston model, measuring distance as a surface embedded in $\mathbb{R}^3$, we will call these lines *Thurston lines*. This definition fits in well with the Burger-Starbird problem, as a line on a "flattened section" of the model would be a geodesic. Our definition of Thurston lines will not include all geodesics. The reader is encouraged to think about complications that occur when geodesics lie in sections of the model that cannot be flattened.

We begin by defining some key terms. A *model triangle* will denote one of the Euclidean triangles. Two model triangles are *adjacent* if they share an edge (meaning they have been glued together along an edge). A *model vertex* is a vertex of any model triangle. Intuitively, a Thurston line will have two properties: It never passes through a model vertex, and when it passes through two adjacent triangles, its restriction to the union of the triangles is a Euclidean line segment, as in FIGURE 3. These properties guarantee that a Thurston line lies in a section of the model that can be flattened.

We now formally define a *Thurston line* to be a set of points ℓ such that

1. The restriction of ℓ to any model triangle T is either empty or a line segment of T containing a no vertex of T.
2. If T_1 and T_2 are adjacent triangles sharing edge $\overline{AB}$, with $\ell \cap \overline{AB} = C$, $X_i \neq C$ and $X_i \in \ell \cap T_i$ for $i = 1, 2$, then $\angle X_1CA \cong \angle X_2CB$.

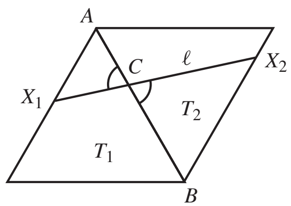

Figure 3 The Thurston line segment $\overline{X_1X_2}$ is the restriction of a Thurston line ℓ to adjacent triangles T_1 and T_2

A *Thurston angle* is now defined naturally as an angle formed by two intersecting Thurston lines. Since the rays of a Thurston angle are subsets of Thurston lines, the vertex of a Thurston angle is *not* a model vertex. Thus, any Thurston angle agrees locally with a Euclidean angle that is inside either a model triangle or two adjacent model triangles, and we define the measure of a Thurston angle to be its Euclidean measure. Define a *Thurston triangle* as the figure bounded by three Thurston lines.

The Burger-Starbird question can now be rephrased as asking us to draw a Thurston triangle with at least one model vertex in its interior and then find the sum of its angles. Curiously, at most two model vertices can lie in the interior of a Thurston triangle, as we will show.

Drawing large Thurston triangles We now turn to the question of how "big" triangles in our geometry can be, by which we mean how many model vertices they may contain. Suppose first that we have a Thurston triangle in our geometry. That is, we have points A, B, and C such that each of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ lies on a piece of the space that can be flattened.

The model triangles partition the interior of $\triangle ABC$ into a collection of complete model triangles and pieces of model triangles. As some of these pieces may be quadrilaterals, we further triangulate the pieces by adding additional edges (but no new vertices). This gives a triangulation of $\triangle ABC$ in which every triangle lies on a flat region of the model, and all the vertices are either model vertices in the interior, or non-model vertices on the boundary. Since all of the triangles in the triangulation are Euclidean, they must each have angle sum of π radians.

To count these triangles, we use Euler's formula for a triangulation of a topological disk: $V - E + F = 1$, where V is the number of vertices, E the number of edges, and F the number of faces in the triangulation. We can write $V = 3 + b + m$, counting the three points A, B, and C, the b additional vertices on the edges $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, and the m internal model vertices. A standard combinatorial argument shows that the total number of edges in the triangulation is

$$E = \frac{3F + b + 3}{2}.$$

Substituting this into Euler's Formula and solving for F yields

$$F = 1 + b + 2m.$$

Since every triangle has an angle sum of π, the sum of all the angles in the triangulation is $\pi F = \pi(1 + b + 2m)$. On the other hand, the angles around each boundary vertex (excepting A, B, and C) sum to π and the angles around each model vertex sum to $7\pi/3$. So we have two ways of computing the sum of the angles in the triangulation, producing the equation

$$\pi(1 + b + 2m) = \angle A + \angle B + \angle C + \pi b + \frac{7\pi}{3}m.$$

Therefore

$$\angle A + \angle B + \angle C = \pi\left(1 - \frac{m}{3}\right),$$

and we have established the following proposition.

PROPOSITION 1. *Any Thurston triangle $\triangle ABC$ has an angle sum equal to $\pi(1 - m/3)$ radians, where m denotes the number of model vertices in the interior of $\triangle ABC$.*

Since any Thurston triangle must have angles with positive measure, it follows that any Thurston triangle can have at most two model vertices on its interior. A triangle containing two model vertices is shown in FIGURE 4.

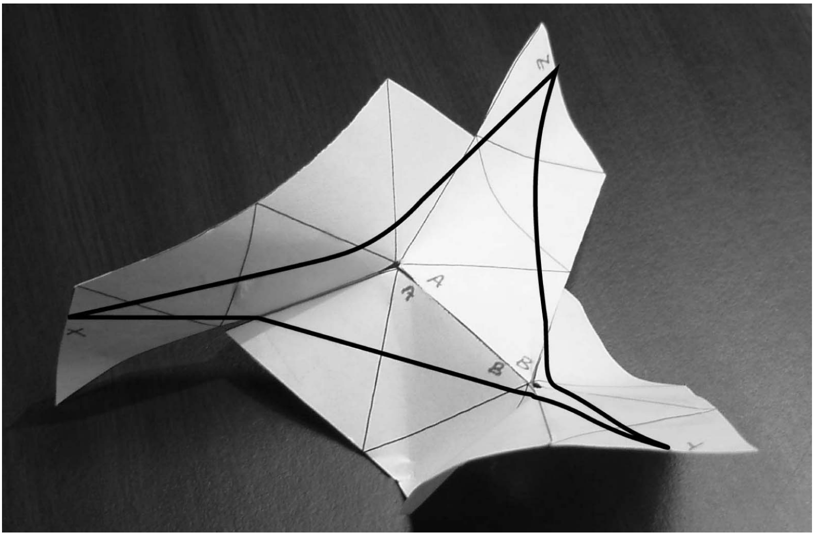

Figure 4 A triangle in the Thurston model enclosing two model vertices

A mapping between the Poincaré and Thurston models Although Thurston lines allow us to get a feel for the curvature of hyperbolic space, they are actually not hyperbolic lines. To define actual hyperbolic lines on the Thurston model, we define a map to the model from one of the standard models of hyperbolic space. We will use the standard Poincaré disk model for the hyperbolic plane, where the geodesics are the diameters of the disk and the circular arcs that are perpendicular to the boundary of the disk. We note that one can tile the Poincaré disk with equilateral triangles so that each angle measures $2\pi/7$, as shown in FIGURE 5. There is a one-to-one correspondence between this tiling and the triangles of the Thurston model. We can use this correspondence to define a bijective mapping from the Poincaré model of hyperbolic space to the Thurston model. The details of this mapping are given in the next two paragraphs for the interested reader, but only the fact of its existence is required for the rest of the paper.

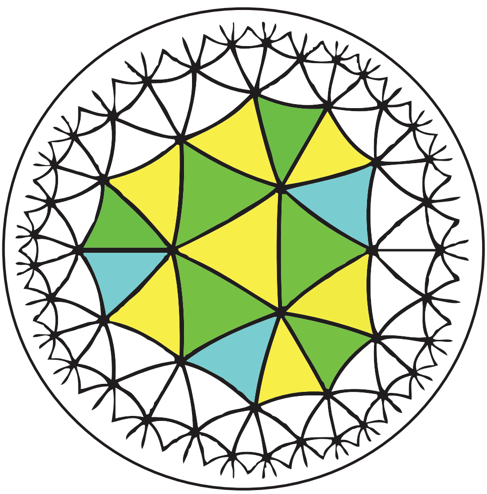

Figure 5 The Poincaré model tiled with equilateral triangles

First, let S be the triangle centered at the origin in the triangulation of the Poincaré disk shown in FIGURE 5. Next, pick a base triangle T in the Thurston model. We define a mapping $f : S \to T$, starting with the Beltrami-Klein disk model of hyperbolic geometry shown in FIGURE 6, where the geodesics are the Euclidean lines in the disk [**2**, pp. 297–301]. We view both the Poincaré model and the Beltrami-Klein model as unit disks in $\mathbb{C}$. Then the function $p(z) = 2z/(1 + |z|^2)$ maps the Poincaré disk to the Beltrami-Klein disk and takes S to a Euclidean equilateral triangle T' centered at the origin of the Beltrami-Klein disk. We map T' to T via a linear rescaling $l(z) = kz$, where k is a positive real constant. It is easy to verify that both p and l are invariant under conjugation by any symmetry of an equilateral triangle. Thus the mapping $f : S \to T$ defined by $f = l \circ p$ is also invariant under these symmetries. Note that the mapping f takes hyperbolic line segments in S to Euclidean line segments in T.

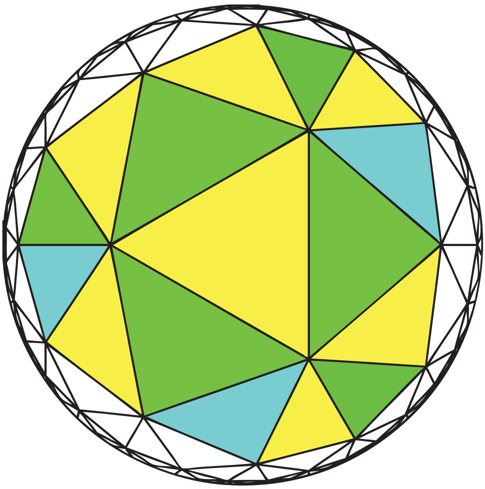

Figure 6 The Beltrami-Klein model tiled with equilateral triangles

We now describe how to extend f to a mapping from the entire Poincaré disk to the Thurston model. Given a triangle S_i of the tiling of the Poincaré disk, there exists an isometry g of the disk such that $g(S_i) = S$. We construct g by choosing a path of triangles from S_i to S in the triangulation, and composing reflections across the sides of the triangles along the path. In the Thurston model, we can inductively reverse this path of triangles and reflections to construct a mapping $\tilde{g}^{-1}$ from T to a unique triangle T_i in the Thurston model. For a point x of S_i, we define $\phi(x) = \tilde{g}^{-1} \circ f \circ g(x)$. To see that this is well defined, observe that if g' were constructed using a different path from S_i to S, then g and g' differ by a symmetry of the equilateral triangle S (and similarly for $\tilde{g}$ and $\tilde{g'}$). Since f is invariant under these symmetries, ϕ is independent of the choice of the path.

We have now defined our mapping ϕ between the models. Under this mapping we have a natural set of lines in the Thurston model, namely the images of hyperbolic lines under ϕ. These lines are only piecewise linear and may pass through model vertices. While, on the face of it, the segments of these hyperbolic lines inside model triangles are Euclidean line segments, the Euclidean distance between two points on the segment is *not* the same as the hyperbolic distance.

However, there is a particular class of these lines that we will call *special hyperbolic lines*, which are also geodesics in the Thurston model. The intersection of a special hyperbolic line with a model triangle is either a side of the triangle or the Euclidean line segment from a vertex to the midpoint of the opposite side. When these special hyperbolic lines pass through a model vertex, by symmetry there is the same Euclidean angle sum (of $7\pi/6$) on either side. FIGURE 7 shows that these special hyperbolic lines arise naturally in the barycentric subdivision of the tiling of the hyperbolic plane by equilateral triangles. We call the triangles of this subdivision the *barycentric triangles*.

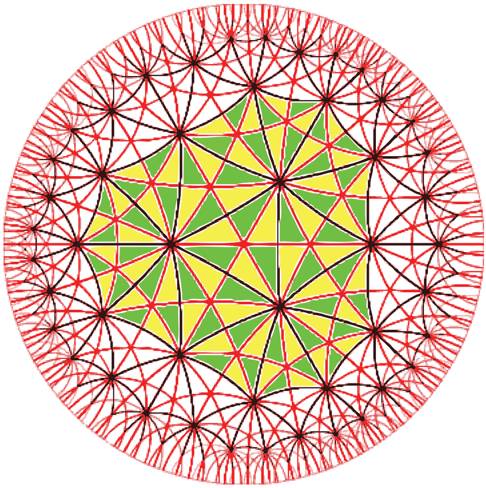

Figure 7 The barycentric subdivision of the Poincaré model, showing the special hyperbolic lines

Drawing large special hyperbolic triangles We have answered Burger and Starbird's question for Thurston triangles, but what if we take a triangle whose edges lie on special hyperbolic lines? Such a triangle is the image of a hyperbolic triangle and, unlike our earlier candidate for a large triangle, can have both internal and boundary model vertices. Triangulate this hyperbolic triangle so that the interior of each small triangle lies inside a model triangle. In this case, again, the angle measure around model vertices in the interior is $7\pi/3$. The angle sum around model vertices on the boundary, however, is only half as much, $7\pi/6$. As in Proposition 1, we discover the following:

PROPOSITION 2. *Any special hyperbolic triangle has angle sum equal to* $\pi(1 - m/3 - n/6)$ *radians, where* m *denotes the number of model vertices in the interior of the triangle and* n *denotes the number of model vertices on the edges of the triangle (not including the triangle vertices themselves).*

Since the smallest angle we could realize on a special hyperbolic triangle has measure $\pi/6$, the proposition implies that the largest number of model vertices that could lie on the triangle is 3 (with $m = 0$ and $n = 3$), and this can be realized, as shown in FIGURE 8.

Deflections of hyperbolic lines in the Thurston model

We have defined *hyperbolic lines* in the Thurston model as the images of the geodesics in the Poincaré model; however, aside from the special hyperbolic lines, we have not discussed what these lines look like in our collection of taped-together triangles. As we mentioned before, the image of a hyperbolic line in any model triangle it passes

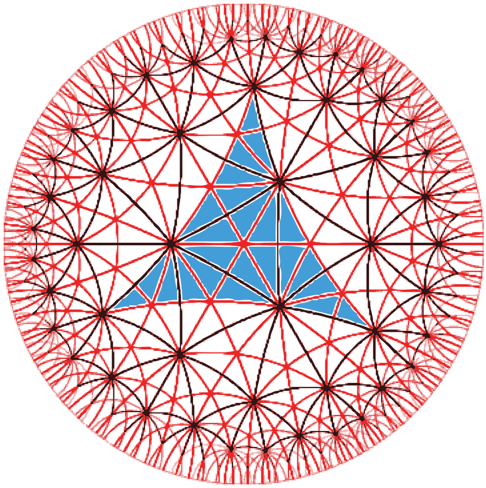

Figure 8 A special hyperbolic triangle in the Poincaré model with three model vertices on the boundary

through is a Euclidean line segment, so the question is how the line bends as it passes between adjacent triangles.

Consider the two equilateral triangles in the Poincaré model on the left in FIGURE 9, together with the hyperbolic line l, and the image of the triangles and the line under ϕ in the Thurston model on the right.

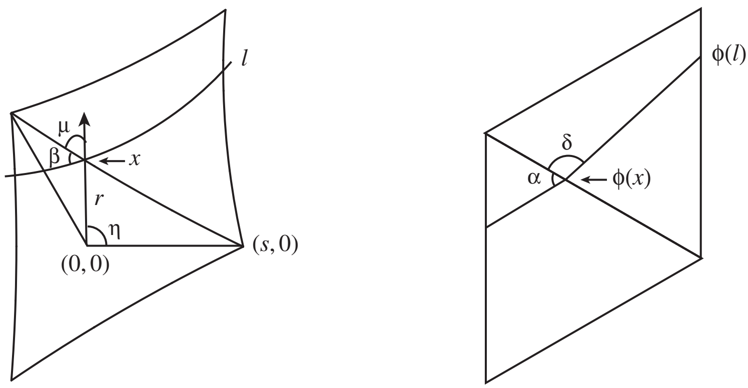

Figure 9 Equilateral triangles and a hyperbolic line in the Poincaré model, and their images in the Thurston model

The angles α and δ in the Thurston model are determined by the angles η and β in the Poincaré model, which together are enough to determine where in the Poincaré disk the hyperbolic line intersects the side of the triangle, as well as the angle of intersection. The formulas for α and δ are quite complicated, involving the derivatives of the mappings ϕ and ϕ^{-1} at the point of intersection. We content ourselves with showing these quantities graphically and leave the (somewhat lengthy) details as an exercise for the reader, with brief answers posted at the MAGAZINE website.

In general, $\alpha + \delta \neq \pi$; we want to measure the *deflection* $\alpha + \delta - \pi$. A graph showing the deflection as a function of the angles η and β appears in FIGURE 10, where η ranges from 0 to $2\pi/3$ and β range from 0 to π.

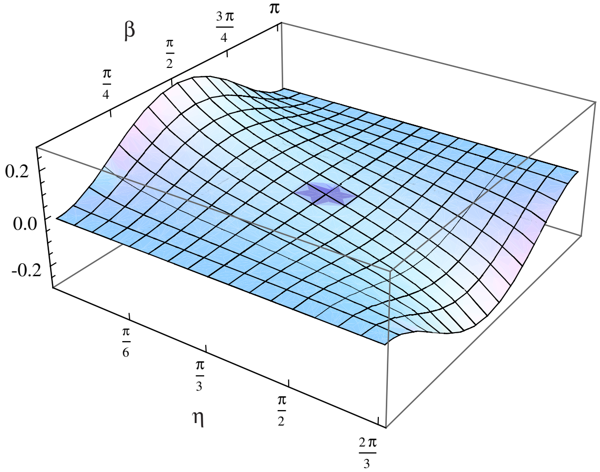

Figure 10 The deflection as a function of the angles η and β

We can now make several interesting observations. First of all, the greatest deflection occurs at $\beta = \pi/2$, when the line is perpendicular to the side of the triangle. FIGURE 11 shows the cross-section of the graph in FIGURE 10 with $\beta = \pi/2$.

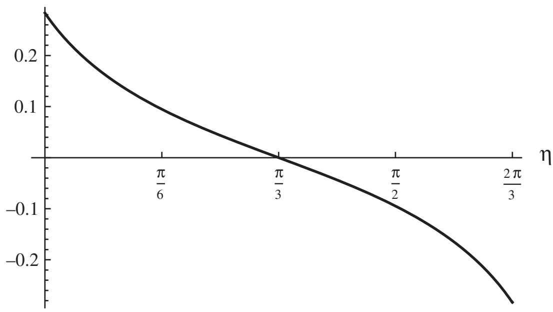

Figure 11 The deflection when $\beta = \pi/2$

The figure shows that, as we approach a vertex, the line is deflected *toward* that vertex, with a maximal deflection that approaches 0.283278 radians (about 16.23 degrees). The amount of the maximal deflection is determined by the equilateral triangle we choose in the Poincaré disk; for the computations that led to FIGURE 11, we chose the triangle centered at the origin with angles measuring $\pi/7$. If we had chosen a larger equilateral triangle (decreasing the angle measures), then this maximal deflection would increase. For example, if the three angle measures were $\pi/8$, the maximal deflection would be 0.469475 radians (about 26.9 degrees). As the angle measures decrease, the number of triangles around each vertex in the corresponding Thurston model increases, and the maximal deflection increases asymptotically toward $\pi/3$. The reason is that, as the number of triangles around each vertex increases, a line passing near one of the vertices will have to pass through more triangles. In the corresponding Thurston model, this means the line will need to be deflected to bend around the vertex. At a deflection of $\pi/3$, a line could be bent into a spiral around a vertex that passes through *all* the triangles around that vertex.

The other interesting observation is that there is *no* deflection when the line passes through the midpoint of a side (when $\eta = \pi/3$). So we see that the Thurston line in the Thurston model that connects the midpoints along a strip of triangles is also a true hyperbolic line, meaning that it is the image of a hyperbolic line under the mapping from the Poincaré model to the Thurston model.

We can also show that there is no deflection through midpoints directly by symmetry considerations. Consider two adjacent model triangles $\triangle ABC$ and $\triangle BCD$, and let x be the midpoint of the shared edge BC, as shown in FIGURE 12. There is an isometry g of the Poincaré model that takes $\phi^{-1}(\triangle ABC)$ to $\phi^{-1}(\triangle DCB)$ by rotating by π radians around $\phi^{-1}(x)$. Consider a point p on the edge AC and its image $q = \phi g \phi^{-1}(p)$. In the Poincaré model, g preserves lines through $\phi^{-1}(x)$; since it exchanges $\phi^{-1}(p)$ and $\phi^{-1}(q)$, the three points $\phi^{-1}(p)$, $\phi^{-1}(x)$ and $\phi^{-1}(q)$ must lie on the same line in the Poincaré model. The image of this line in the Thurston model is the pair of line segments px and xq. However, since g is an isometry, we know that $|px| = |qx|$, $|Cx| = |Bx|$ and $|pC| = |qB|$, so by Side-Side-Side congruence the triangles $\triangle pxC$ and $\triangle qxB$ are congruent. In particular, $\angle pxC = \angle qxB$, which means that the image of the hyperbolic line is the Euclidean line between p and q. We conclude that there is no deflection through the midpoint x.

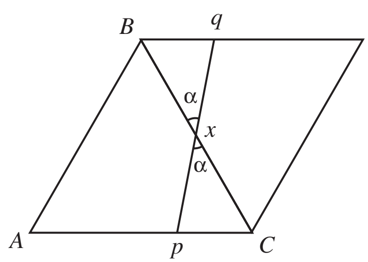

Figure 12 The image $\overline{pq}$ of a hyperbolic line segment through the midpoint x

FIGURE 13 compares hyperbolic lines and Thurston lines in a segment of the Thurston model. In each example, we have drawn both the Thurston line and the hyperbolic line connecting two points in the Thurston model. We can see that when the hyperbolic line is near the midpoints, it is almost straight and very close to the Thurston line; however, when it is farther from the midpoints, the deflections are much greater.

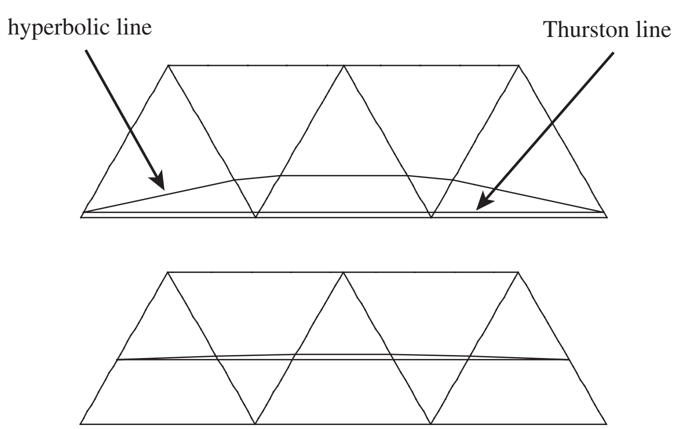

Figure 13 Thurston lines and hyperbolic lines in the Thurston model

Pick's Theorem in Thurston's model

We continue to explore our model by establishing a hyperbolic analog of Pick's Theorem, which gives a simple formula in Euclidean geometry for computing the area of a polygon drawn on a unit square lattice (meaning that the area of one square of the lattice is 1). It has many applications and generalizations [**4, 3**]. Here is the simplest form of Pick's Theorem: If a polygon P is drawn on a square lattice so that all the vertices are lattice points, if there are i vertices inside the polygon, and if there are b vertices on the boundary of the polygon, then the area of the polygon is

$$A(P) = i + b/2 - 1.$$

For example, the area of the polygon in FIGURE 14 is $A = 5 + 7/2 - 1 = 7.5$.

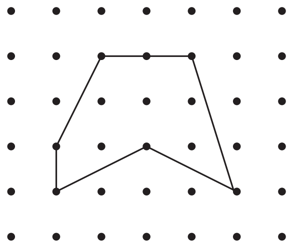

Figure 14 A polygon in a unit square lattice whose area is 7.5 units by Pick's Theorem

The special hyperbolic lines of the Thurston model are the lines corresponding to the barycentric subdivision of our triangulation of hyperbolic space. Notice that all the small triangles formed by this subdivision are congruent, and so they all have the same area α. Now, suppose we have a special hyperbolic figure R in the model, that is, each side of R is made up of special hyperbolic lines and each vertex is either a model vertex, a model center or a model midpoint as in FIGURE 15. We will also assume that R is simply connected and hence a topological disk.

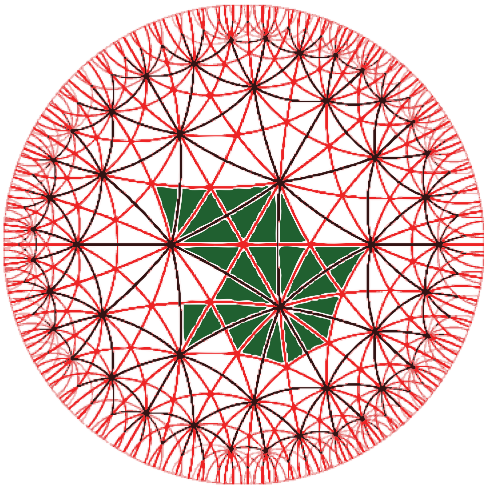

Figure 15 A special hyperbolic figure R (shaded) in the Poincaré model whose area is 27 units by Proposition 3

The area of R is equal to α times the number of barycentric triangles contained in R. So, if we want the area of R, we can count the number of barycentric triangles in R. For this, we once again recall Euler's formula for a tiling of a disk: $V - E + F = 1$. Letting our lattice points be the centers, midpoints, and vertices of the model triangles (so the lattice points are the vertices of the barycentric subdivision), we know that each internal edge lies on exactly two faces, whereas each boundary edge lies on exactly one face. Our faces are all barycentric triangles, so every face is bordered by three edges. Letting V_b be the number of boundary vertices, E_i be the number of internal edges, and E_b be the number of boundary edges, we have $E_b = V_b$ and $2E_i + E_b = 3F$, or $3F = 2E - E_b = 2E - V_b$. Thus $E = (1/2)(3F + V_b)$. Letting V_i denote the number of internal vertices, we have

$$V - E + F = (V_b + V_i) - (1/2)(3F + V_b) + F = 1.$$

Solving for F we obtain

$$F = 2V_i + V_b - 2.$$

But F is the number of barycentric triangles we have in the region R. Consequently, we have proved:

PROPOSITION 3. *Let R be a region bounded by special hyperbolic lines in the Thurston model. Then the area of region R is given by*

$$\mathit{Area}(R) = (2V_i + V_b - 2)\alpha,$$

where α is the area of the barycentric triangle.

In fact, as others have noted, the hardest step in proving Pick's theorem is to show that any minimal triangle has area $1/2$, and the result follows from Euler's formula. In our case, all minimal triangles are congruent, since they are images of the fundamental domain for the group action on the hyperbolic plane, so our result is not too surprising. Notice that in FIGURE 15, the region has 5 internal vertices and 19 boundary vertices, so the area is $(10 + 19 - 2)\alpha = 27\alpha$; and indeed the region contains 27 triangles of the barycentric subdivision.

Suppose we make a slightly different restriction on our region R, namely that all the vertices must be model vertices. One quickly sees that there are only two minimal triangles in this case, the model triangle and a triangle created by bisecting the quadrilateral formed by two adjacent model triangles. By symmetry arguments, both of these triangles have area $\beta = 6\alpha$. As a result, we have:

PROPOSITION 4. *Let R be a region bounded by special hyperbolic lines in the Thurston model, and let V_i denote the number of model vertices inside R and V_b denote the number of model vertices on the boundary of R. Then the area of region R is given by*

$$\mathit{Area}(R) = (2V_i + V_b - 2)\beta,$$

where β is the area of the model triangle.

General Thurston models

Of course, Thurston's model is just one way to model hyperbolic space; there are many others that may allow some constructions to be performed more easily. It turns out that if we generalize the Thurston model, then we can make flatter models that allow for a wider variety of triangles. We define a *general Thurston model*: Take any regular triangulation of hyperbolic space given by an integer triple (n_1, n_2, n_3), representing the fundamental triangle with angle measures $(2\pi/n_1, 2\pi/n_2, 2\pi/n_3)$. We make one additional requirement that if one of n_1, n_2, n_3 is odd, then the other two are equal. With these conditions, we can create a tiling of hyperbolic space with the property that all angles about any vertex are congruent. Associated to this tiling, we take a Euclidean triangle with angle measures a_1, a_2, and a_3 such that $n_1 a_1 = n_2 a_2 = n_3 a_3$. Then we tape together n_1 vertices of angle measure a_1, n_2 vertices of angle measure a_2, and n_3 angles of measure a_3. In the standard Thurston model, $n_1 = n_2 = n_3 = 7$ and $a_1 = a_2 = a_3 = \pi/3$.

The requirement $n_1 a_1 = n_2 a_2 = n_3 a_3$ means that at each vertex of the model, the excess in angle is the same. This means that the amount the paper must bend in order for us to tape together the triangles is the same at each vertex; we might naively refer to this as the *curvature*. In these general models you lose a little bit of regularity in the sense that the vertices are not evenly spaced out, and it also becomes a little harder to flatten out the space to draw a straight line. On the other hand, you can make the curvature much smaller, allowing you to draw a greater variety of triangles. A simple calculation shows that under these conditions

$$a_i = \frac{n_j n_k}{n_1 n_2 + n_2 n_3 + n_1 n_3}\pi,$$

and that the excess angle glued around a vertex (called the *angle excess*) is

$$\mathrm{E}(n_1, n_2, n_3) = \left(\frac{n_1 n_2 n_3}{n_1 n_2 + n_2 n_3 + n_1 n_3} - 2\right)\pi.$$

Of course, if $1/n_1 + 1/n_2 + 1/n_3 > 1/2$, we have too little angle around a vertex and our triangle corresponds to a tiling of the sphere, so that $(3, 3, 3)$ produces a tetrahedron, $(4, 4, 4)$ produces an octahedron, and $(5, 5, 5)$ produces an icosahedron. If $1/n_1 + 1/n_2 + 1/n_3 = 1/2$, then our triangle tiles Euclidean space. Thus, for our purposes, we will restrict our attention to the case where $1/n_1 + 1/n_2 + 1/n_3 < 1/2$. Noting that $E(n_1, n_2, n_3)$ is increasing in each n_i, to find the minimal excess we can simply check the smallest possible triples satisfying our conditions, namely $(6, 6, 7)$, $(5, 8, 8)$, $(4, 8, 10)$, and $(4, 6, 14)$. The table below shows the excess for each of these, along with the standard Thurston model $(7, 7, 7)$.

(n_1, n_2, n_3)	$E(n_1, n_2, n_3)$
$(7, 7, 7)$	$\pi/3$
$(6, 6, 7)$	$\pi/10$
$(5, 8, 8)$	$2\pi/9$
$(4, 8, 10)$	$2\pi/19$
$(4, 6, 14)$	$2\pi/41$

Thus, the smallest excess comes with the choice $(4, 6, 14)$, shown in FIGURE 16 (FIGURE 17 gives a schematic you can copy to construct your own model). Although this initially looks quite different from the Thurston model, it actually arises from its barycentric subdivision. That is, this triangle corresponds to the minimal triangle we saw before in the Thurston model!

The arguments that we gave before for the standard Thurston model carry over directly to the new model, thus we have

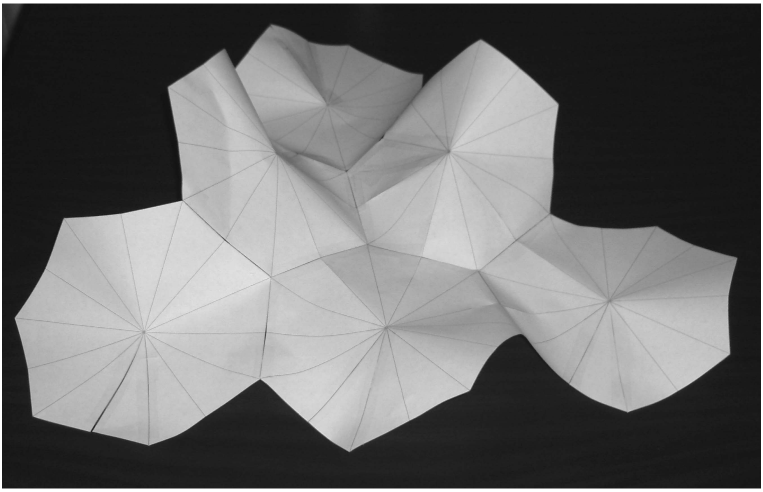

Figure 16 The (4, 6, 14) general Thurston model

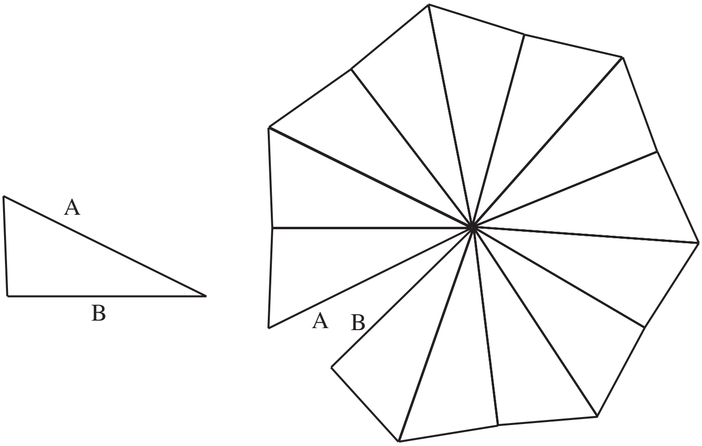

Figure 17 Schematic for the (4, 6, 14) general Thurston model, with the 14th triangle around the vertex to be pasted in along edges A and B

PROPOSITION 5. *In the general Thurston (n_1, n_2, n_3) space, any Thurston triangle $\triangle ABC$ has angle sum equal to $\pi - E(n_1, n_2, n_3)V_i$ where V_i denotes the number of model vertices in the interior of $\triangle ABC$. Moreover, if we take a special hyperbolic triangle with model vertices, then the angle sum is $\pi - (1/2)E(n_1, n_2, n_3)(2V_i + V_b - 3)$ where V_i is the number of model vertices lying in the interior of the triangle and V_b is the number of model vertices on the boundary (including the vertices of the triangle).*

So, in the (4, 6, 14)-model it is possible to draw a Thurston triangle containing as many as 20 model vertices.

We also have an analog of Pick's theorem for the general models:

PROPOSITION 6. *Let R be a region bounded by (n_1, n_2, n_3)-model hyperbolic lines, and again let V_i denote the number of internal model vertices and V_b denote the number of model vertices on the boundary of R. Then the area of region R is given by*

$$Area(R) = (2V_i + V_b - 2)\beta,$$

where β is the area of the model triangle.

Gauss-Bonnet Theorem

We can put Propositions 5 and 6 together to get a special case of the Gauss-Bonnet formula, one of the most important theorems in differential geometry. Recall that the Gauss-Bonnet formula states that the area A of a triangle in a surface of constant curvature κ is given by the formula

$$-\kappa A = (\pi - a_1 - a_2 - a_3),$$

where a_1, a_2, and a_3 denote the measurements of the interior angles of the triangle. We will derive a similar formula relating the area and angle sum of a special hyperbolic triangle in the (n_1, n_2, n_3)-model, whose vertices are all model vertices. This is particularly useful for the (4, 6, 14)-model (or any model where n_1, n_2, n_3 are all distinct), where *all* special hyperbolic triangles have model vertices.

From Proposition 6, the area of a special hyperbolic triangle with model vertices is $A = (2V_i + V_b - 2)\beta$, where β is the area of the model triangle. On the other hand, by Proposition 5, the sum of the angles a_1, a_2, and a_3 of the triangle is given by

$$\begin{aligned} a_1 + a_2 + a_3 &= \pi - \frac{1}{2}E(n_1, n_2, n_3)(2V_i + V_b - 3) \\ &= \pi - \frac{1}{2}E(n_1, n_2, n_3)\left(\frac{A}{\beta} - 1\right) \\ &= \pi - \frac{E(n_1, n_2, n_3)}{2\beta}(A - \beta). \end{aligned}$$

It now follows that we can write the area A in terms of the angles of the special hyperbolic triangle, the area of a model triangle, and the angle excess $E(n_1, n_2, n_3)$. Specifically,

$$A = \beta + \frac{2\beta}{E(n_1, n_2, n_3)}(\pi - a_1 - a_2 - a_3),$$

where a_1, a_2, and a_3 are the measures of the angles of the triangle. Here the curvature of the model is approximated by the angle excess $E(n_1, n_2, n_3)$, which corresponds to our observation that reducing the angle excess results in a flatter model.

We can derive a formula even closer to the Gauss-Bonnet formula by introducing a new variable α_i, defined below (for brevity, we let $E = E(n_1, n_2, n_3)$):

$$a_i = \left(1 + \frac{E}{2\pi}\right)\alpha_i \qquad \text{or} \qquad \alpha_i = \frac{a_i}{1 + \frac{E}{2\pi}}.$$

Then our expression for the area becomes

$$\begin{aligned} A &= \beta + \frac{2\beta}{E}(\pi - a_1 - a_2 - a_3) \\ &= \beta + \frac{2\beta}{E}\left(\pi - \left(1 + \frac{E}{2\pi}\right)(\alpha_1 + \alpha_2 + \alpha_3)\right) \\ &= \beta\left(1 + \frac{2\pi}{E} - \left(\frac{2}{E} + \frac{1}{\pi}\right)(\alpha_1 + \alpha_2 + \alpha_3)\right) \end{aligned}$$

$$= \beta \left(\frac{2}{E} + \frac{1}{\pi} \right) (\pi - \alpha_1 - \alpha_2 - \alpha_3)$$

$$= \beta \left(\frac{2\pi + E}{\pi E} \right) (\pi - \alpha_1 - \alpha_2 - \alpha_3).$$

We have proved the following analogue of the Gauss-Bonnet Theorem:

PROPOSITION 7. *Consider a special hyperbolic triangle with model vertices in the general Thurston (n_1, n_2, n_3)-model, where β is the area of a model triangle. Say that the triangle has area A and angles a_1, a_2, a_3. Then*

$$-\kappa A = (\pi - \alpha_1 - \alpha_2 - \alpha_3),$$

where

$$\alpha_i = \frac{a_i}{1 + \frac{E}{2\pi}}, \qquad \kappa = -\frac{\pi E}{\beta(2\pi + E)} \quad \text{and} \quad E = E(n_1, n_2, n_3).$$

How can we interpret α_i and κ? If we consider the preimage of our special hyperbolic triangle in the Poincaré model of hyperbolic space (as described earlier), then the preimage of an angle a at one of the model vertices (with angle excess $E = E(n_1, n_2, n_3)$) is exactly

$$\alpha = a \frac{2\pi}{2\pi + E} = \frac{a}{1 + \frac{E}{2\pi}}$$

This means that α_i is just the true hyperbolic angle corresponding to the angle a_i at a model vertex.

To understand κ, consider the preimage of a model triangle in the hyperbolic surface of constant curvature -1. This triangle has angles $2\pi/n_1, 2\pi/n_2, 2\pi/n_3$, so by the classical Gauss-Bonnet Theorem, its area γ is

$$\begin{aligned}
\gamma &= -\left(\frac{2\pi}{n_1} + \frac{2\pi}{n_2} + \frac{2\pi}{n_3} - \pi \right) \\
&= \pi - \left(\frac{n_1 n_2 + n_2 n_3 + n_1 n_3}{n_1 n_2 n_3} \right) 2\pi \\
&= \pi - \frac{\pi}{E + 2\pi} 2\pi \\
&= \frac{E\pi + 2\pi^2 - 2\pi^2}{E + 2\pi} = \frac{E\pi}{E + 2\pi}.
\end{aligned}$$

Then $\kappa = -\gamma/\beta$ measures the ratio of the area of the preimage of the model triangle in the surface with constant curvature -1 to the area of the model triangle. As the model triangle gets larger (so the model is flatter), κ will get closer to 0, so κ is a reasonable measure of the curvature of the model. Moreover, as the angle excess E shrinks, κ will also get smaller. This means that the (4, 6, 14)-model, with the smallest angle excess, gives a significantly flatter model, in which it is easier to follow the hyperbolic lines and illustrate the Gauss-Bonnet theorem.

In a college geometry class, we have used these models to introduce students to curvature and the Gauss-Bonnet theorem, without any of the difficult differential geometry required to prove the full Gauss-Bonnet Theorem. Students can be led to discover for themselves one of the greatest theorems of mathematics, starting from no more than paper triangles and tape!

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Summary In looking at a common physical model of the hyperbolic plane, the authors encountered surprising difficulties in drawing a large triangle. Understanding these difficulties leads to an intriguing exploration of the geometry of the Thurston model of the hyperbolic plane. In this exploration we encounter topics ranging from combinatorics and Pick's Theorem to differential geometry and the Gauss-Bonnet Theorem.

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