ARTICLES

Geometry, Voting, and Paradoxes

DONALD G. SAARI
Northwestern University
Evanston, IL 60208–2730

FABRICE VALOGNES
CREME, Université de Caen
Caen
France

1. Problems

What could be easier than "voting?" After all, to vote we just count how many people favor each candidate. What can go wrong with something so elementary as this?

Actually, a lot. As mathematicians and others have shown over the last two centuries, once there are at least three candidates—not an atypical situation—the winner need not be whom the voters really want. Such bad outcomes may occur not only because some voters continue to vote long after death; bad outcomes can also be caused by hidden mathematical peculiarities.

We illustrate with an example from **[6]**, where fifteen people select a common beverage from among M (Milk), B (Beer), and W (Wine). If " $\succ$ " means "is preferred to" and if the voters' preferences are as follows:

Number	Preference
6	$M \succ W \succ B$
5	$B \succ W \succ M$
4	$W \succ B \succ M$

(1)

then the plurality outcome (where each person votes for his or her favorite beverage) is $M \succ B \succ W$ with the 6:5:4 tally. Apparently, Milk is the beverage of choice.

Before ordering a keg of Milk, let's pause. Is Milk truly the voters' beverage of choice? If so, we would expect voters to prefer Milk to Beer. But as the next table shows, these voters actually prefer Beer to Milk:

Number	Preferences	Milk	Beer
6	$M \succ W \succ B$	6	0
5	$B \succ W \succ M$	0	5
4	$W \succ B \succ M$	0	4
	Total	6	9

Similarly, 9 voters prefer Wine to Milk and 10 prefer Wine to Beer. This creates a contradiction and potential controversy among the party goers, because these pairwise comparisons suggest that the voters really prefer $W \succ B \succ M$, the *ranking opposite to the plurality outcome*. What went wrong?

Mathematicians This type of problem, coupled with the obvious importance of elections, motivated several eighteenth century mathematicians to investigate the mathematical peculiarities of elections. The mathematician J. C. de Borda was probably the first to consider these issues from an academic perspective when, in 1770, he questioned whether the French Academy of Science was electing to membership whom they really wanted. His concern, as illustrated by the beverage example, is that the "winner" of the widely used plurality vote can be the candidate the voters view as "inferior."

Borda [1] devised an alternative procedure, now called the *Borda Count*, which assigns 2, 1, and 0 points, respectively, to a voter's top, middle, and bottom-ranked candidate; candidates are then ranked according to the sum of assigned points. To see that this method can change the outcome, consider the Borda Count tally for the beverage example:

Number	Preferences	Milk	Beer	Wine
6	$M \succ W \succ B$	6×2	0	6×1
5	$B \succ W \succ M$	0	5×2	5×1
4	$W \succ B \succ M$	0	4×1	4×2
	Total	12	14	19

(2)

This produces the $W \succ B \succ M$ outcome, which agrees with the pairwise election rankings.

The Borda Count appears to be the "correct" voting procedure—at least for this example. But what happens in general? Are there examples of sets of voters' preferences, called *profiles*, for which the Borda Count does poorly? Why not use other weights, such as $(6,5,0)$ or $(4,1,0)$, instead of Borda's choice of $(2,1,0)$? Tallying methods that assign a specified number of points to a voter's first, second, and third ranked candidate are called *positional voting methods*. When normalized to assign a single point to a voter's top-ranked candidate, the point assignment defines a *voting vector* $\mathbf{w}_\lambda = (1, \lambda, 0)$, $0 \leq \lambda \leq 1$. For instance, the normalized forms of $(6,5,0)$ and the Borda Count are, respectively, $\mathbf{w}_{\frac{5}{6}} = (\frac{6}{6}, \frac{5}{6}, 0)$ and $\mathbf{w}_{\frac{1}{2}} = (1, \frac{1}{2}, 0)$. Because $\mathbf{w}_1 = (1,1,0)$ effectively requires a voter to vote against his or her bottom-ranked candidate, it is called the *antiplurality method.*

The $\mathbf{w}_\lambda$ normalization makes it clear that there is a continuum of tallying methods where each is characterized by the weight (the λ-value) placed on a voter's second-ranked candidate. Faced with all these possibilities, it was only natural for Borda's mathematical colleagues, such as Laplace, Condorcet, and others, to question which $\mathbf{w}_\lambda$ method is optimal in the sense that its outcomes best reflect the views of the voters. The debate they started continues today.

Condorcet Marie-Jean-Antoine-Nicolas de Caritat Condorcet, the French mathematician, philosopher, and politician, added to the controversy in the 1780's by arguing that, instead of using a $\mathbf{w}_\lambda$ method, the outcomes should be decided strictly in terms of the pairwise vote. The *Condorcet winner* is the candidate who beats all other candidates in pairwise elections. With the preferences of table (1), Wine, which wins a majority vote over each of the other beverages, is the Condorcet winner. Milk is the *Condorcet loser*.

Until recently the Condorcet winner was almost universally accepted as the ultimate choice. (See [**6**, **7**, **8**] for arguments questioning this concept.) But, it has problems. To illustrate just one difficulty, suppose a mathematics department uses pairwise voting to choose a calculus book from among the choices $\{A, B, C\}$. A natural way to select the book is by elimination, where after comparing two choices, say $\{A, B\}$, the winner is compared with the remaining choice, C. Suppose the views of the department members are

Number	Preferences
5	$A \succ B \succ C$
5	$B \succ C \succ A$
5	$C \succ A \succ B$

(3)

As the following table shows, A wins the initial $\{A, B\}$ comparison only to be beaten by C. In both elections the winner wins with a landslide two-thirds of the vote, so it seems safe to declare that the departmental ranking is the decisive $C \succ A \succ B$.

Number	Preference	A	B	A	C
5	$A \succ B \succ C$	5	0	5	0
5	$B \succ C \succ A$	0	5	5	0
5	$C \succ A \succ B$	5	0	0	5
	Totals	10	5	5	10

Although the outcome appears to be unquestionable, let's question it. We already know that C beats A and A beats B, so it remains to determine whether "top-ranked" C beats "bottom-ranked" B. We might expect no surprises, but there is one: B beats C by the same two-thirds landslide vote. In other words, this profile defines the *cyclic* election outcomes

$$A \succ B, \qquad B \succ C, \qquad C \succ A,$$

whereby whichever candidate is voted upon last, wins—decisively. In particular, there is no Condorcet winner or loser.

Condorcet understood that cycles could arise from pairwise voting; he demonstrated this behavior by introducing the example of table (3). Such an example is now known as a *Condorcet profile*.

Cycles, then, make it impossible to select an "optimal" candidate. (For a companion discussion of the problems of cycles, see [**9**].) But elections are intended to decide, so competing approaches have been devised to avoid stalemates. For instance, A. Copeland, a mathematician from the University of Michigan, developed a method which is similar to how hockey teams are ranked. A competing procedure, which involves counting the number of transpositions needed to convert one ranking into another, was devised by the mathematician J. Kemeny, from Dartmouth. (For a geometric analysis of both approaches, see [**10**, **11**].)

Complexity and geometry Which method is best? Although this issue appears straightforward, progress has been seriously hindered by the complexity of the combinatorics. A traditional way to compare procedures is to construct profiles that show how one method has a failing not suffered by another. But to construct examples, we need to determine how many voters must be of each type so that the resulting election outcomes capture the desired phenomenon.

To illustrate the complexity of the combinatorics, we offer some challenges. For instance, can the Condorcet and Borda winners differ? If so, find an illustrating profile. The beverage example proves that different positional methods create different election outcomes. Is there a general description explaining how election results change with changes in the $\mathbf{w}_\lambda$ methods? When using different $\mathbf{w}_\lambda$ voting vectors to tally ballots in the profile of table (1), either Wine, Milk, or both always emerges as the top choice (see **[6]**). Are there voters' profiles where *each* candidate is the "winner" for an appropriate $\mathbf{w}_\lambda$? Are the supporting examples isolated or robust? Can we characterize *all* possible examples? What is the minimum number of voters needed to create each election oddity?

In recent years, progress has been made on these concerns by replacing the traditional combinatoric method with a geometric perspective. A summary of this "geometry of voting" approach for three candidates is in the textbook **[6]**, while progress for any number of candidates (obtained by use of symmetry groups, etc.) is reported in **[7, 8]**. In this essay we demonstrate how geometry dramatically reduces these previously complicated issues into forms simple enough to be presented to students who can graph elementary algebraic equations.

2. Voter Types

A voter's "type" is defined by how the voter strictly ranks the candidates $\{A, B, C\}$. For convenience, denote these types by the following numbers:

Type	Preference	Type	Preference
1	$A \succ B \succ C$	4	$C \succ B \succ A$
2	$A \succ C \succ B$	5	$B \succ C \succ A$
3	$C \succ A \succ B$	6	$B \succ A \succ C$

(4)

These types are reflected in the geometry of the equilateral triangle of FIGURE 1, where each candidate is identified with a vertex. Each point in the triangle is assigned an ordinal ranking of the candidates according to how close the point is to each vertex

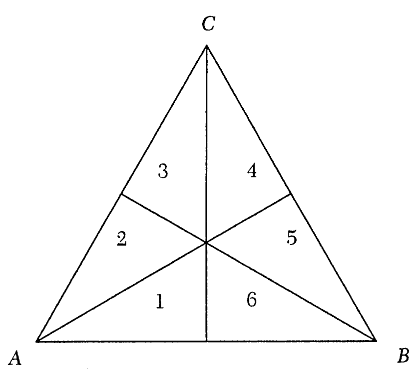


FIGURE 1
The representation triangle and ranking regions.

where, as in love, "closer is better." Points on the vertical line, for instance, are equidistant from A and B, so all of them are *indifferent* between these options; this is denoted by $A \sim B$. Similarly, all points in the triangular sector "1" are closest to A, next closest to B, and farthest from C, and so define the $A \succ B \succ C$ ranking.

Considerable insight and unexpected conclusions already arise when the voters' beliefs are restricted to only three specified preference types. This is what we discuss here. But selecting three of six voter types creates $\binom{6}{3} = 20$ situations to examine. Fortunately, as shown in Section 5, symmetry arguments reduce the number to three.

3. Condorcet Examples

The mystery of the pairwise voting cycles justifies starting with the setting where voters' preferences come from the three types involved in the Condorcet profile of table (3). This setting is captured in FIGURE 2a, where the three preference types define a symmetric "pinwheel" configuration. (This "$\mathbb{Z}_3$ orbit" symmetry causes the cycles.)

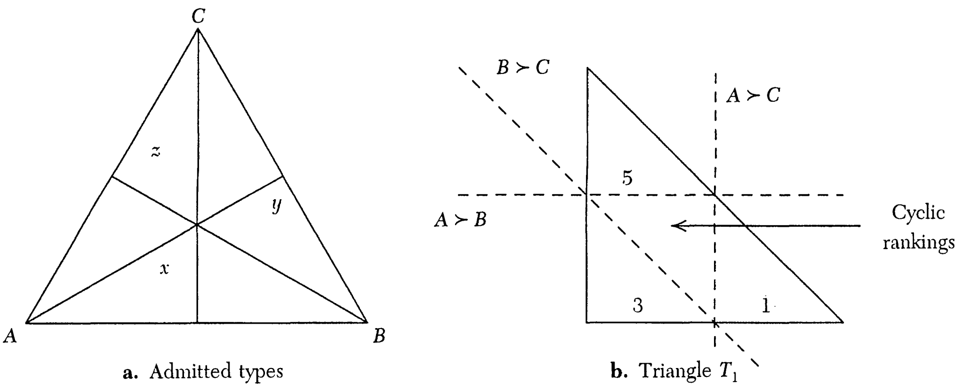


FIGURE 2
Condorcet example setting.

If n_j is the number of voters of type j, then the total number of voters is $n_1 + n_3 + n_5 = n$. Instead of dealing with integers, we divide by n, so that $x = n_1/n$, $y = n_5/n$, and $z = n_3/n$ represent the fractions of all voters that are of each type. In the textbook example, for instance, $x = y = z = \frac{5}{15}$.

The constraint $x + y + z = 1$, or $z = 1 - (x + y)$, allows us to represent all possible profiles as the (rational) points of the triangle

$$T_1 = \{ (x, y) \mid x, y \geq 0, x + y \leq 1\}$$

of FIGURE 2b. (The origin is at the lower left corner.) For a point $(x, y) \in T_1$, the fraction of all voters with type 1 and 5 preferences are given, respectively, by the x and y values; the fraction of all voters with a type 3 preference is $1 - x - y$.

Pairwise outcomes One hindrance to our understanding of election behavior is the difficulty of associating profiles with their election outcomes. With geometry, however, this reduces to graphing elementary algebraic equations. In an $\{A, B\}$ election, for instance, it follows from FIGURE 2a that only a type 5 voter votes for B; all other voters are on the A side of the $A \sim B$ line, so they vote for A. Therefore, B beats A if and only if $y > x + z = x + (1 - x - y)$, or if $y > \frac{1}{2}$. The T_1 boundary for this region is the horizontal dashed line of FIGURE 2b.

The analysis for the remaining two pairs is similar. For an $\{A, C\}$ election, it follows from FIGURE 2a that only type 1 voters prefer $A \succ C$, so A beats C if and only if $x > \frac{1}{2}$; the boundary is the vertical dashed line of FIGURE 2b. Likewise with $\{B, C\}$: candidate C wins if and only if $z = 1 - (x + y) > \frac{1}{2}$, or if $(x + y) < \frac{1}{2}$; the T_1 boundary is the slanted dashed line in FIGURE 2b.

As it is easy to determine which pairwise outcomes occur on each side of each dashed T_1 boundary line, we know which election rankings are associated with each of the four resulting regions of profiles. For instance, the region to the extreme right, with T_1 vertex $(1, 0)$, is on the $A \succ B$, $A \succ C$, $B \succ C$ sides of the boundary lines, so all of these profiles define the type 1 ranking $A \succ B \succ C$. Similarly, two of the other regions identify all profiles resulting in type 3 or type 5 pairwise outcomes. Our real interest is in the remaining small triangle in the center, which identifies all profiles that cause cyclic pairwise outcomes.

To illustrate how to use this geometry, suppose we want to determine the minimum number of voters required to construct examples for any of the admitted outcomes. To do so, notice that n, the total number of voters, is a common denominator for x and y. The answer, then, just involves finding in each region the points (x, y) with the smallest common denominator.

As all points (x, y) with common denominator 2 are either vertices of T_1 or vertices of the small triangle that causes cyclic outcomes, all two-voter examples have either unanimity outcomes, or *non-transitive rankings involving tie votes*. To illustrate, point $(\frac{1}{2}, 0)$ defines the rankings $A \sim C$, $C \sim B$, even though $A \succ B$. (So, peculiar election outcomes already arise with only two voters.) With three voters, $(\frac{1}{3}, \frac{1}{3})$ is in the center of the cyclic region. (Point $(\frac{1}{3}, \frac{1}{3})$ corresponds to modifying table (3) to have only one voter of each type.) Similar arguments show that points on the boundary lines require four voters. Therefore, with no more than four voters, we can create examples of all admitted pairwise rankings.

One of the many oddities of voting theory is how conclusions can depend upon whether the number of voters is odd or even. The geometry shows that this peculiarity is caused by how rational points are distributed within a region, depending on the parity of the smallest common denominator. We illustrate by raising another question: Can cycles occur if only one voter in a large population has type 3 preferences? With n voters, this condition requires $z = 1/n$, so a required (x, y) point must satisfy $x + y = 1 - 1/n$ and be in the cyclic region near $\left(\frac{1}{2}, \frac{1}{2}\right)$. If n is even, the only choices of $\left(\frac{n-2}{2n}, \frac{1}{2}\right)$ or $\left(\frac{1}{2}, \frac{n-2}{2n}\right)$ are not admissible because they are boundary points. Thus, this particular behavior occurs if and only if n is odd and $x = y = \frac{n-1}{2n}$.

Probabilities There is a large literature in which complicated techniques are used to compute the probabilities of various election outcomes. (See, for instance, the excellent bibliography **[4]**.) With geometry, however, it is easy to compute the likelihood of each outcome. For instance, if each point (i.e., each profile) in T_1 is equally likely, then the common areas of the four regions prove that each outcome occurs with probability $\frac{1}{4}$. Similarly, say that a profile probability is *centrally distributed* if the likelihood of profile (p_1, p_2, p_3) is the same as (p_2, p_1, p_3), or of any of the four other ways these p_j values can be permuted. An example is the multinomial distribution. This symmetry over voter types means that with a centrally distributed profile probability, all three transitive outcomes are equally likely. By appealing to the central limit theorem, we identify a wide class of settings where the likelihood of cyclic rankings dominates.

These $\frac{1}{4}$ probability values represent *limits* as the number of voters becomes very large. To explain with n voters, notice that the number of fractions x and y with common denominator n that satisfy $x+y=0$ (so $z=1$) is the number of admissible numerators for x; it is 1. Similarly, if $n-j$ of the n voters have type 3 beliefs (so $z=1-j/n$), the number of points (x, y) satisfying $x+y=j/n$ is $j+1$. The standard identity

$$\sum_{j=1}^{k} j=\binom{k+1}{2}=\frac{k(k+1)}{2} \tag{5}$$

ensures that there are $\binom{n+2}{2}$ rational points in T_1 with common denominator n. Therefore, n voters create $\binom{n+2}{2}$ different profiles among these three beliefs.

An important observation (illustrated with $n=2,3$) is that these $\binom{n+2}{2}$ points need not be equally distributed among the four regions. So, to compute the number of points (or profiles) in each region, notice that the points in the small triangle defining cyclic outcomes are those (x, y) with $x<1/2$, $y<1/2$, and $x+y>\frac{1}{2}$. For odd values of n, j different (x, y) points in the cyclic region satisfy $x+y=1-\frac{j}{n}=1-z$, $j=2,\dots,(n-1)/2$. Using equation (5), this total of $\frac{(n-1)(n+1)}{8}$ points means that the fraction of the T_1 points in the cyclic region is

$$\frac{(n-1)(n+1)}{4(n+1)(n+2)}=\frac{1}{4}\left(1-\frac{3}{n+2}\right);$$

this tends to $\frac{1}{4}$ as $n \to \infty$. Similarly, for even values of n we have the smaller

$$\frac{1}{4}\left(1-\frac{9n-6}{(n+1)(n+2)}\right) \to \frac{1}{4}.$$

The following theorem results from similarly easy computations.

THEOREM 1. *When voters are restricted to types* 1, 3, *and* 5, *the four possible strict pairwise outcomes include these three types and the cyclic rankings* $A \succ B \succ C \succ A$. *If profile points in* T_1 *are assumed to be centrally distributed, then the three transitive rankings are equally likely. In the case of n voters, and we assume that all points in* T_1 *are equally likely, the probability of strict rankings with cyclic outcomes is* $\frac{1}{4}\left(1-\frac{3}{n+2}\right)$ *if n is odd and* $\frac{1}{4}\left(1-\frac{9n-6}{(n+1)(n+2)}\right)$ *if n is even. The likelihood of a strict transitive ranking is* $\frac{1}{4}\left(1+\frac{1}{n+2}\right)$ *if n is odd and* $\frac{1}{4}\left(1-\frac{1}{n+1}\right)$ *if n is even.*

While the $\frac{1}{4}$ probabilities are rapidly approached as the number of voters increases, notice the strikingly different values that occur for small n-values. For instance, with $n=3$, instead of approximately $\frac{1}{4}$ of the points in the cyclic region, there are only $\frac{1}{10}$ of them. For $n=4$, this probability drops to zero, then rebounds to $\frac{1}{7}$ for $n=5$ only to drop to $\frac{1}{28}$ for $n=6$. Again, this oddity involving the parity of n reflects the distribution of rational points in T_1.

Positional outcomes The geometry also identifies all possible conflicts between the pairwise and the $\mathbf{w}_\lambda$ outcomes. Using FIGURE 1 to compute candidate B's $\mathbf{w}_\lambda=(1,\lambda,0)$ tally of an election, notice that she receives one point from each voter who has her top-ranked; these voters are of types 5 and 6, where B is a vertex of the ranking regions. With our FIGURE 2a restriction, B receives $y \times 1$ points. The second place votes of λ points per voter come from the adjacent 1 and 4 regions of FIGURE 1. With FIGURE 2a, this adds λx points for B. As the remaining two regions (2 and 3) represent where B is bottom-ranked, they contribute no points, so the total tally is $y + \lambda x$. The $\mathbf{w}_\lambda$ tallies for all candidates are as follows:

Candidate	Tally
A	$(-\lambda)x - \lambda y + \lambda$
B	$y + \lambda x$
C	$1 - x + (\lambda - 1)y$

(6)

The rest of the analysis mimics what we did with the pairwise vote. Namely, to determine which profiles define the relative $A \succ B$ or $B \succ A$ rankings, plot the $A \sim B$ boundary line defined by equating the A and B tallies. This defines the parametrized family of equations $(1 - 2\lambda)x - (1 + \lambda)y + \lambda = 0$. Because $x = \frac{1}{3}$, $y = \frac{1}{3}$ satisfies this equation for all λ-values, all of these lines pass through $(\frac{1}{3}, \frac{1}{3})$, which we call the *rotation point*. The line defined by λ is determined by the rotation point and $\left(\frac{-\lambda}{1-2\lambda}, 0\right)$, its x-intercept. The results for all candidate pairs follow:

Pair	Equation	Rotation Pt	x-axis Pt
$A \sim B$	$(1 - 2\lambda)x - (1 + \lambda)y = -\lambda$	$(\frac{1}{3}, \frac{1}{3})$	$\left(\frac{-\lambda}{1-2\lambda}, 0\right)$
$A \sim C$	$(2 - \lambda x + (1 - 2\lambda)y = 1 - \lambda$	$(\frac{1}{3}, \frac{1}{3})$	$\left(\frac{1-\lambda}{2-\lambda}, 0\right)$
$B \sim C$	$(1 + \lambda)x + (2 - \lambda)y = 1$	$(\frac{1}{3}, \frac{1}{3})$	$\left(\frac{1}{1+\lambda}, 0\right)$

(7)

The effects of these lines are depicted in FIGURE 3 for three special cases: the plurality vote ($\lambda = 0$); the Borda Count ($\lambda = \frac{1}{2}$); and the antiplurality method ($\lambda = 1$). This figure identifies interesting behavior because it displays how election outcomes change with the procedure. To explain, notice that although the three boundary lines for the $\lambda = 0$ and $\lambda = 1$ triangles agree, each line is identified with a different pair of candidates. Connecting them is a fascinating rotation where, as the value of λ increases, each boundary line rotates in a clockwise direction from its $\lambda = 0$ setting to reach the adjacent boundary line position when $\lambda = 1$. For instance, the $A \sim C$

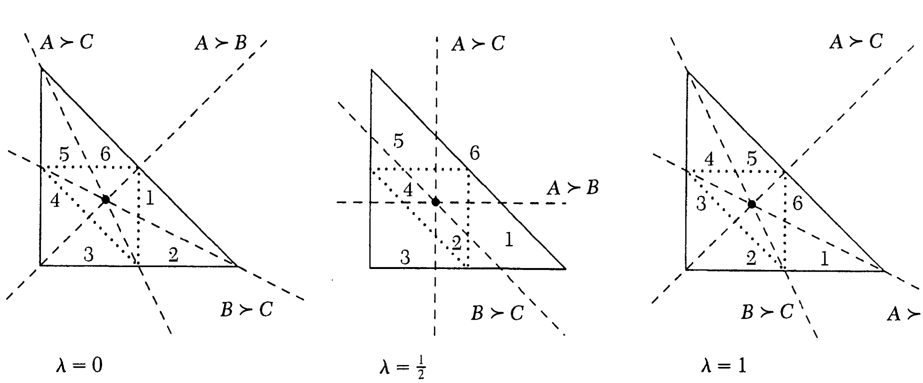


FIGURE 3
Computing $\mathbf{w}_\lambda$ outcomes.

boundary line passes through the (0, 1) vertex of T_1 when $\lambda = 0$ (the plurality vote), becomes vertical when $\lambda = \frac{1}{2}$ (the Borda Count), and stops at what had been the $A \sim B$ original position when $\lambda = 1$.

An immediate consequence of this rotation is that, with the exception of the $(\frac{1}{3}, \frac{1}{3})$ point (the Condorcet profile where all $\mathbf{w}_\lambda$ methods have a completely tied outcome), each profile experiences *three different* $\mathbf{w}_\lambda$ *election rankings as* λ *varies through its admissible values*. If a point is on a boundary line when $\lambda = 0$, then two of the rankings have ties and one is strict. Otherwise, two of the rankings are strict and one involves a pairwise tie. The geometry shows that, rather than being an isolated phenomenon, conflict is unavoidable.

As a second consequence, consider a region with transitive pairwise votes; say, the region labeled "1" in FIGURE 2b. (In FIGURE 3, this set of profiles is the region to the right of the vertical dotted line.) By examining this region in the $\lambda = 0$ and $\lambda = 1$ triangles, we see that these profiles allow two different strict plurality and antiplurality election outcomes. For instance, the pairwise $A \succ B \succ C$ outcome is accompanied by a plurality ranking of either $A \succ B \succ C$ (type 1) or the conflicting $A \succ C \succ B$ (type 2). While the difference in outcomes creates a conflict, at least the plurality and pairwise procedures agree on which candidate is top-ranked. A similar analysis holds for the antiplurality $\lambda = 1$ where the conflicting ranking is $B \succ A \succ C$ (type 6). Here, however, the pairwise and antiplurality methods agree only on who should be bottom-ranked; they can disagree on the rest of the ranking and who should win.

The Borda Count allows not only two but *three* strict rankings for profiles from each of the three strict pairwise ranking regions. In fact, the rotation of the indifference lines and the monotonicity of the x coordinate (of the "x-axis point" in table (7) proves that for each $\lambda \in (0, 1)$, $\mathbf{w}_\lambda$ *admits three different strict election rankings for each of the three sets of profiles*. This, of course, provides plenty of robust examples of conflict between the pairwise and $\mathbf{w}_\lambda$ rankings.

The triangle defining cyclic pairwise outcomes admits even more conflict: here, anything can happen with any $\mathbf{w}_\lambda$ method. Namely, accompanying a pairwise cycle, we can have any strict $\mathbf{w}_\lambda$ ranking, any $\mathbf{w}_\lambda$ ranking with one pair tied, or a completely tied outcome.

Because (from elementary trigonometry) all ranking regions of the $\lambda = 0$ and $\lambda = 1$ triangles have the same area, each has the (limiting) probability of $\frac{1}{6}$. This is also true for the smaller triangle with cyclic pairwise voting. Consequently in either case—whether we consider all profiles in T_1 or restrict attention to profiles causing pairwise cycles—the limiting probability for any strict ranking for the $\lambda = 0, 1$ procedures is $\frac{1}{6}$. The Borda Count ($\lambda = \frac{1}{2}$) favors the three outcomes of types 1, 3, and 5 (the types from the profile) with limiting probability of $\frac{2}{9}$; the remaining three types have limiting probabilities of $\frac{1}{9}$. What connects these different values is that (from the x-axis values of table (7)) the areas of some regions monotonically decrease, while others increase, as $\lambda \to \frac{1}{2}^-$. Then they change to monotonically approach the common value $\frac{1}{6}$ as $\lambda \to 1$. These statements, and others are equally easy to verify, are collected in the following theorem:

THEOREM 2. *If the three voter types* 1, 3, *and* 5 *are allowed, then each profile that is not a Condorcet profile admits three different* $\mathbf{w}_\lambda$ *election outcomes as* λ *varies.*

The set of profiles with pairwise votes that define a particular strict transitive outcome allows only two strict election rankings with the plurality and with the antiplurality vote. In each case, one of these outcomes agrees with the pairwise rankings. All other $\mathbf{w}_\lambda$ *outcomes admit three different strict rankings, one of which agrees with the pairwise ranking. The profile set causing cyclic pairwise outcomes admits all possible* $\mathbf{w}_\lambda$ *rankings.*

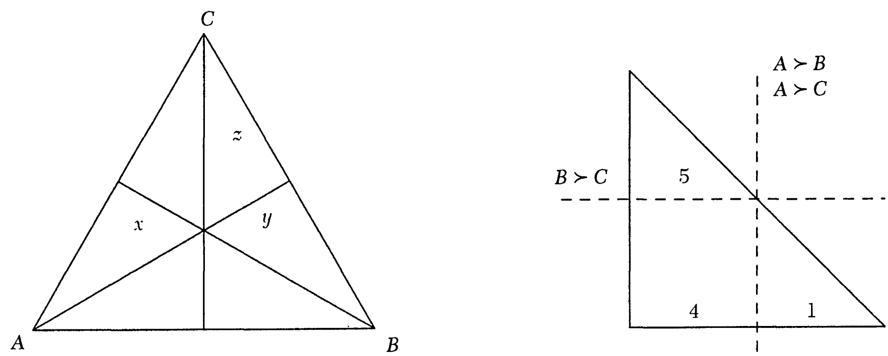
If all T_1 points are equally likely, then the limiting probability of any strict election ranking (in either the set of all profiles or the cyclic region) is $\frac{1}{6}$ for $\lambda = 0, 1$. For the Borda Count the limiting probability for either setting is $\frac{2}{9}$ for outcomes of types 1, 3, and 5, and $\frac{1}{9}$ for the remaining three types.

The likelihood of an election outcome being of a particular type either strictly increases or strictly decreases as $\lambda \to \frac{1}{2}$.

These results show that even with only three types of voter preferences, conflict can arise among the pairwise and positional election outcomes. So, which procedure is "best?" Frankly, the answer is not clear from this information. For instance, the fact that the plurality and pairwise outcomes identify the same candidate as being top-ranked can be fashioned into a strong argument in favor of the plurality vote—at least for this setting. On the other hand, the ranking of a unanimity profile should be its election ranking, so we should expect election outcomes to favor the three particular types represented in the profile. This is true for the Borda Count, but only to a lesser degree for the other $\mathbf{w}_\lambda$ methods. This observation can be developed into an argument supporting the Borda Count. With a little imagination, an argument can probably be fashioned to support any other procedure. So which procedure should we use?

4. The Beverage Example Revisited

While the Condorcet setting allows profiles to have different $\mathbf{w}_\lambda$ outcomes, the conflict is nowhere near as spectacular as that displayed in the beverage example, where completely reversed $\mathbf{w}_\lambda$ election rankings occur for different λ values. This example, where two of the preferences share an edge of the FIGURE 1 triangle and the third ranking is from a ranking region with the remaining vertex, captures a familiar election setting where one candidate, A, is favored (top-ranked) by a portion of the voters, but strongly opposed (bottom-ranked) by the rest of them. The voters who dislike A, however, split in their opinions about the other two candidates. (This may have been the situation created by the candidacy of P. Buchanan during the 1996 Republican Presidential primaries.) As in FIGURE 4a, define $x = n_2/n$, $y = n_5/n$, and $z = n_4/n$. To connect the beverage example with FIGURE 4a, identify M, B, W respectively with A, B, C so that beverage profile of equation (1) becomes $x = \frac{6}{15}$, $y = \frac{4}{15}$, and $z = \frac{5}{15}$.



a. Admitted types **b.** Triangle T_1

FIGURE 4
The beverage example setting.

Again, the $z = 1-(x+y)$ restriction allows all possible profiles to be represented as (rational) points in the FIGURE 4b triangle $T_2 = \{(x, y) \mid x, y \geq 0, x+y \leq 1\}$.

Pairwise outcomes Just as in Section 3, identifying profiles with their accompanying pairwise outcomes involves only elementary algebra. As FIGURE 4a shows, in an $\{A, B\}$ election only type 2 voters vote for A, so A beats B if and only if $x > \frac{1}{2}$. Similarly, in an $\{A, C\}$ election, A beats C if and only if $x > \frac{1}{2}$. The common T_2 boundary for these conditions is the vertical dashed line of FIGURE 4b. For the remaining pair $\{B, C\}$, B wins if and only if $y > \frac{1}{2}$; here the T_2 boundary is the horizontal dashed line in FIGURE 4b.

The pairwise election combinations allow only three (strict) transitive pairwise ranking outcomes; no real surprises occur with the pairwise vote. The election rankings are denoted in FIGURE 4b with the voter type numbers. Again, by assuming that each T_2 point is equally likely, the areas of these regions show that the pairwise outcomes define the type 4 ranking $C \succ B \succ A$ (of the beverage example) with limiting probability $\frac{1}{2}$, and each of the other two types with limiting probability $\frac{1}{4}$. Again, elementary computations using equation (5) show that these limiting values are approached with order $1/n$.

Positional outcomes This setting's particular interest is in the conflict among the pairwise and $\mathbf{w}_\lambda$ outcomes. As in Section 3, the $\mathbf{w}_\lambda$ tally for each candidate is as follows:

Candidate	Tally
A	x
B	$y + \lambda z = (1-\lambda)y - \lambda x + \lambda$
C	$z + \lambda(x+y) = 1-(1-\lambda)(x+y)$

(8)

By setting pairs of tallies equal to each other, the $\mathbf{w}_\lambda$ outcomes change according to the following table of parametrized equations.

Pair	Equation	Rotation Pt	x-axis Pt
$A \sim B$	$(1+\lambda)x-(1-\lambda)y = \lambda$	$(\frac{1}{2}, \frac{1}{2})$	$\left(\frac{\lambda}{1+\lambda}, 0\right)$
$A \sim C$	$(2-\lambda)x+(1-\lambda)y = 1$	$(1, -1)$	$\left(\frac{1-\lambda}{1-2\lambda}, 0\right)$
$B \sim C$	$(1-2\lambda)x+2(1-\lambda)y = 1-\lambda$	$(0, \frac{1}{2})$	$\left(\frac{1-\lambda}{1-2\lambda}, 0\right)$

(9)

A major difference from Section 3 is that the rotation point of each line differs with each pair. As we will see, this is what causes new kinds of election outcomes to occur. The boundary lines, and the resulting division of profiles identified with the plurality $(\lambda = 0)$, Borda $(\lambda = \frac{1}{2})$, and antiplurality $(\lambda = 1)$ voting systems, are represented in FIGURE 5. (The three rotation points are indicated by the solid dots.)

These figures immediately disclose all sorts of conflicting election outcomes. For instance, the square defined by the dotted lines are all profiles defining the $C \succ B \succ A$ pairwise ranking. The $\lambda = 0$ portion of FIGURE 5 shows that these pairwise rankings can be accompanied by *any* plurality ranking. In other words, expect conflict; the table (1) example demonstrates only the one possibility of a $A \succ B \succ C$ plurality outcome.

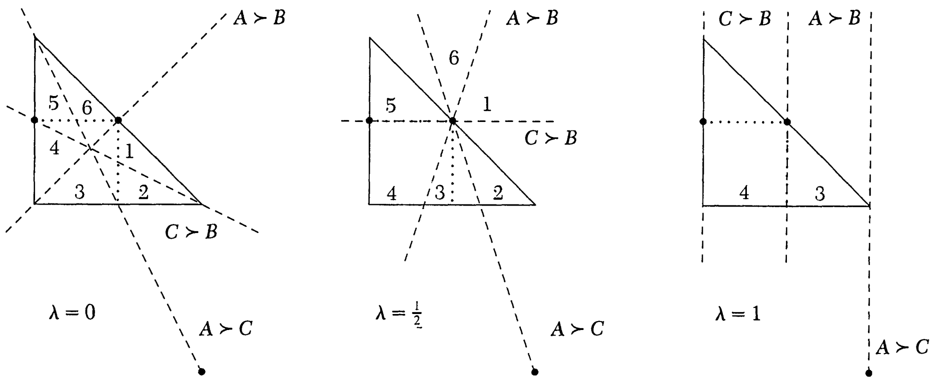


FIGURE 5
Computing $\mathbf{w}_\lambda$ outcomes.

Moreover, it appears from these figures (and we show next why it is true) that the same serious conflict holds for *all* $\mathbf{w}_\lambda$ where $0 \leq \lambda < \frac{1}{2}$.

To find even more fascinating changes, notice the importance of the profile which defines a completely tied $\mathbf{w}_\lambda$ election outcome. By being on the boundary for all $\mathbf{w}_\lambda$ ranking regions, this point identifies how election rankings vary with λ. We already know there are significant changes because for $\lambda = 0$ the point is at the safe $(\frac{1}{3}, \frac{1}{3})$ location (with one voter for each of the three preferences); it moves to the T_2 boundary at $(\frac{1}{2}, \frac{1}{2})$ when $\lambda = \frac{1}{2}$; it vanishes at infinity when $\lambda = 1$. These changes in position are direct consequences of the different locations of the rotation points for each pair.

This observation suggests that important information about election behavior is obtained by plotting how this point of a completely tied election varies with λ. This point is the intersection of the $A \sim B$ and $B \sim C$ boundary surfaces, so, by solving these equations for (x, y) in terms of λ, the equation for this point is

$$(x, y) = \left(\frac{1+\lambda}{3}, \frac{1-\lambda+\lambda^2}{3(1-\lambda)} \right), \qquad 0 \leq \lambda \leq 1, \tag{10}$$

or, because $\lambda = 3x - 1$,

$$y = \frac{1-3x+3x^2}{2-3x} = -x + \frac{1}{3} - \frac{1}{3(3x-2)}.$$

This curve is plotted in FIGURE 6 along with the $\lambda = 0$ boundary lines. The accompanying magnified version shows the translated $\lambda = \frac{1}{4}$ boundary lines.

As FIGURE 6 offers a wealth of information about election behavior, so we describe only what happens to the profiles in the square defined by the dotted lines (with a $C \succ B \succ A$ pairwise ranking); analysis of the other regions is left to the interested reader. First, the fact that the curve approaches infinity as $\lambda \to 1$ is what allows the $\lambda = 1$ figure to have parallel, vertical boundary lines; this is true for no other λ value. Consequently, *for all* $\lambda < 1$, *at least two different* $\mathbf{w}_\lambda$ *strict rankings accompany the* $C \succ B \succ A$ *pairwise outcomes*. Because the point of complete ties leaves T_2 only after the Borda Count, *for* $\lambda < \frac{1}{2}$ *any conflicting* $\mathbf{w}_\lambda$ *ranking can accompany these pairwise rankings*.

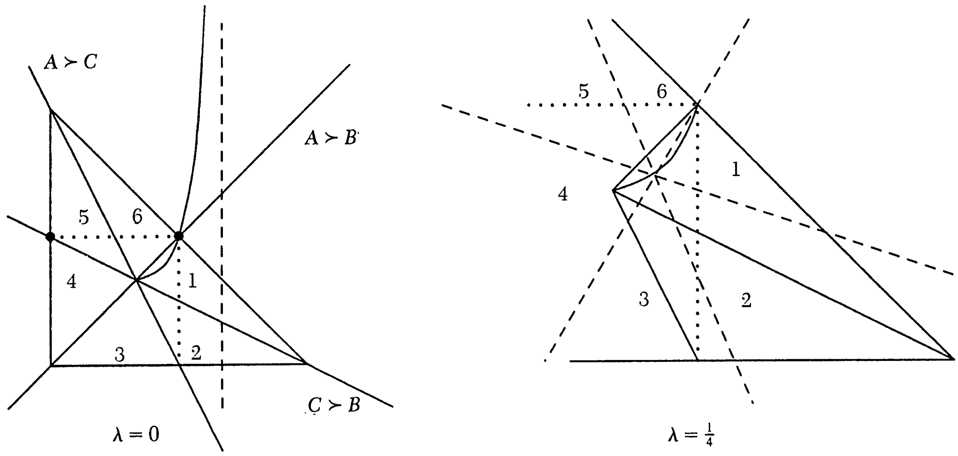


FIGURE 6
Locus of the completely tied points.

This curve also determines how $\mathbf{w}_\lambda$ rankings change with a fixed profile. To indicate the analysis, consider a profile $\mathbf{p}$ located between the curve and the $A \sim B$ plurality line. Although the plurality election ranking for $\mathbf{p}$ is $A \succ B \succ C$, as λ increases in value the $\mathbf{w}_\lambda$ complete tie point moves along the curve forcing different ranking regions to cross $\mathbf{p}$. This can be illustrated with the magnified version of a portion of T_2 in FIGURE 6 which shows the $\lambda = \frac{1}{4}$ regions. If $\mathbf{p}$ has a type 4 election outcome for $\lambda = \frac{1}{4}$, then $\mathbf{p}$ already produced election outcomes of types 1, 6, and 5 for earlier λ values. As table (4) shows, $\mathbf{p}$ has the property that each candidate wins with the appropriate $\mathbf{w}_\lambda$. Furthermore, counting tied outcomes shows that each profile in the region between the curve and the $A \sim B$ plurality boundary line admits *seven different election rankings* for different $\mathbf{w}_\lambda$ procedures. (A similar argument shows that profiles below the curve and with the $A \succ B \succ C$ plurality election outcome have seven rankings where each candidate is bottom-ranked with some $\mathbf{w}_\lambda$.)

The next natural question is to find the smallest number of voters allowing the peculiarity that anyone can be elected. This requires finding a point (x, y) in this region with the smallest possible common denominator. Because (x, y) must satisfy $\frac{1}{3} < x < \frac{1}{2}$ and $y < x$, while being above the curve (so $y > \frac{1}{3}$), we start by seeking a point with least common denominator so that $\frac{1}{3} < y < x < \frac{1}{2}$. This point is $(\frac{5}{11}, \frac{4}{11})$, so examples require at least eleven voters. As the first point above the curve is $(\frac{8}{19}, \frac{7}{19})$, the desired profile involves nineteen voters. It is

Number	Preferences
8	$A \succ C \succ B$
7	$B \succ C \succ A$
4	$C \succ B \succ A$

(11)

where $\lambda \in (\frac{1}{4}, \frac{3}{11})$ ensures the victory of B.

We can find even more. The limiting probability of this peculiar behavior depends on the area between the curve and the $\lambda = 0$ boundary line for $A \sim B$ (that is, the line

$y = x$). This area is

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \left(2x - \frac{1}{3} + \frac{1}{3(3x-2)}\right) dx = \frac{1}{12} - \frac{1}{9}\ln 2.$$

By considering only the profiles in the square (with area $\frac{1}{4}$), the limiting probability is four times this value, or $\frac{1}{3} - \frac{4}{9}\ln 2 \approx 0.0253$.

A small selection of the election behavior attributed to profiles restricted to the "beverage-type" preferences follows.

THEOREM 3. *Suppose the profiles are restricted to preferences from the beverage example. With limiting probability* $\frac{1}{6} - \frac{2}{9}\ln 2$, *it is possible for a profile to elect all three candidates when the ballots are tallied with different* $\mathbf{w}_\lambda$ *methods. The profile must have at least* 19 *voters; the smallest such profile is given in table* (11). *When restricted to where the pairwise votes define the* $C \succ B \succ A$ *ranking, the probability of this behavior is* $\frac{1}{3} - \frac{4}{9}\ln 2$.

The election phenomenon where each candidate is bottom-ranked with some $\mathbf{w}_\lambda$ *procedure has limiting probability* $\frac{1}{6} - [\frac{1}{6} - \frac{2}{9}\ln 2] = \frac{2}{9}\ln 2 \approx 0.1540$. (*When restricted to the profiles with* $C \succ B \succ A$ *pairwise outcomes, the probability is* 0.308.) *All such profiles involve at least nine voters; a nine-voter example results if two voters are removed from each type in table* (1).

For $\lambda = 0$ *the limiting probability of all six possible strict outcomes are equal. For the Borda Count, there are four possible strict outcomes. The limiting probability of a type* 2 *or type* 3 *outcome is* $\frac{1}{6}$, *of a type* 4 *outcome is* $\frac{7}{12}$, *and of a type* 5 *outcome is* $\frac{1}{4}$. *For the antiplurality vote, the limiting probabilities for the type* 3 *and* 4 *outcomes are, respectively,* $\frac{1}{4}$ *and* $\frac{3}{4}$.

5. Symmetry

We have discussed only two of the $\binom{6}{3}$ possible cases. However, by exploiting the symmetry admitted by voting, we have nearly completed the analysis.

Neutrality To introduce the first symmetry, suppose that, for totally unexplained reasons, *everyone* in the beverage example of table (1) confused Beer and Wine. (For instance, a ranking listed as $M \succ W \succ B$ was intended to be $M \succ B \succ W$.) It is easy to correct this mistake: if *all* voters interchanged Wine and Beer on their ballots, then we just interchange Wine and Beer in the election outcomes.

This property, where if every voter permutes the names of the candidates in the same manner, then the election outcome experiences a similar change, is called *neutrality*. More precisely, if σ is a permutation of the names of the candidates, then let $\sigma(\mathbf{p})$ be the profile where these changes occur for each voter in the profile $\mathbf{p}$. Then a voting procedure f satisfies *neutrality* if for any permutation of names σ and for any profile $\mathbf{p}$ we have

$$f(\sigma(\mathbf{p})) = \sigma(f(\mathbf{p})). \tag{12}$$

Neutrality converts our analysis in Section 4 of what happens when voters have types $(2, 4, 5)$ into what happens when voters have types $(1, 4, 5)$. This is because, according to table 1, the second situation is obtained from the first by flipping the triangle about the $B \sim C$ axis. In mathematical terms, by interchanging B and C

names in each ranking of the first setting, we obtain the second one. Thus, the two settings are related by equation (12) and the permutation interchanging B and C.

Other permutations and the resulting settings are listed below. This symmetry and the (2,4,5) prototype account for six of the 20 possibilities.

Setting	Permutation	Setting	Permutation
(2,4,5)	Identity	(1,4,5)	$B\to C, C\to B$
(2,3,5)	$A\to B, B\to A$	(1,3,6)	$A\to C, C\to A$
(2,3,6)	$A\to B, B\to C, C\to A$	(1,4,6)	$A\to C, C\to B, B\to A$

(13)

Similarly, neutrality converts the analysis of Section 3, where voters' preferences come from $\{1,3,5\}$ types, into the setting where voters' preferences come from $\{2,4,6\}$. Here, any transposition, such as $A\to B$, $B\to A$ suffices. This accounts for eight of the 20 cases.

Reversal To introduce the next voting symmetry, suppose for the beverage example of table (1) that each voter misunderstood the instructions and marked the ballots in a completely reversed order. For instance, voters who marked their ballots as $M\succ W\succ B$ really meant $B\succ W\succ M$. If this reversal holds for all voters, then it is reasonable to assume that the election ranking can be corrected by reversing the original one. Namely, if ρ represents the operation of reversing a ranking, it is natural to assume that

$$f(\rho(\mathbf{p}))=\rho(f(\mathbf{p})).$$

The only difficulty with this assumption is that, in general, it is false. To illustrate with the beverage example, apply the plurality vote to the bottom-ranked candidates to discover that, when preferences are reversed, the plurality election outcome *remains* $M\succ B\succ W$, with a 9:6:0 tally.

To discover what does occur with reversal symmetry, recall that the antiplurality vote requires a voter to vote *against* his or her bottom-ranked candidate. Thus, it is equivalent to voting for our bottom-ranked candidate and then reversing the outcome. So, if we apply the plurality vote to $\rho(\mathbf{p})$ and reverse the resulting ranking, we obtain the antiplurality ranking for $\mathbf{p}$. (Readers may wish to carry out this computation with the beverage example of table (1).) The following theorem asserts that the same reversal effect applies more generally.

THEOREM 4. (*See* [6].) *Let* $f(\mathbf{p},\mathbf{w}_\lambda)$ *be the* $\mathbf{w}_\lambda$ *election ranking for profile* $\mathbf{p}$. *All profiles* $\mathbf{p}$ *and positional methods satisfy*

$$f(\mathbf{p},\mathbf{w}_\lambda)=\rho(f(\rho(\mathbf{p}),\mathbf{w}_{1-\lambda})). \qquad (14)$$

Equation (14) allows us to handle six more of the $\binom{6}{3}$ cases. To illustrate what happens, some details are given for what we call the "reversed beverage" example, where the preferences are denoted by FIGURE 7a. As A is top-ranked by two types of voters and bottom-ranked by the remaining type, it is reasonable to expect no election surprises. This is not the case; instead, the election behavior is very similar to that described in Section 4. Indeed, the reason for the similarity of outcomes and the "reversed beverage" nomenclature comes from comparing FIGURE 4a and FIGURE 7a. Each letter x, y, and z is reversed relative to the complete indifference point. We emphasize the consequences of this reversal.

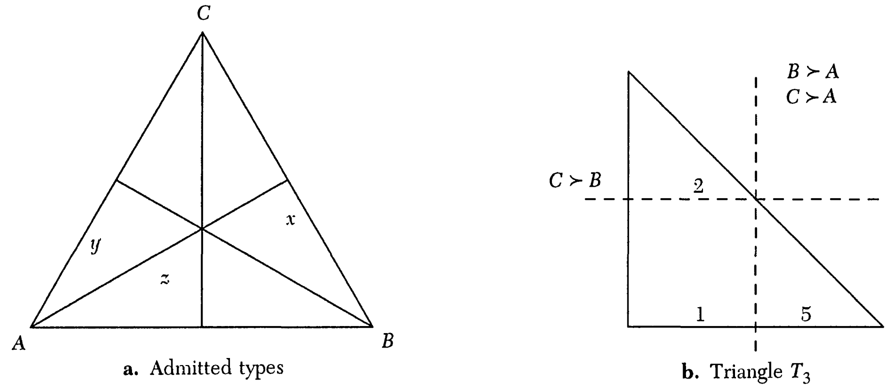


FIGURE 7
The reversed beverage example setting.

One aspect of reversing preference is apparent by comparing FIGURE 4b and 7b: the figures agree, but the rankings are reversed. This reversal continues with the following table, which catalogues information about the $\mathbf{w}_\lambda$ boundary lines:

Pair	Equation	Rotation Pt	x-axis Pt
$A \sim B$	$(2-\lambda)x - \lambda y = 1 - \lambda$	$(\frac{1}{2}, \frac{1}{2})$	$\left(\frac{1-\lambda}{2-\lambda}, 0\right)$
$A \sim C$	$(1+\lambda)x + \lambda y = 1$	$(1, -1)$	$\left(\frac{1}{1+\lambda}, 0\right)$
$B \sim C$	$(1-2\lambda)x - 2\lambda y = -\lambda$	$(0, \frac{1}{2})$	$\left(\frac{-\lambda}{1-2\lambda}, 0\right)$

(15)

To convert table (15) into table (9), let $\mu = 1 - \lambda$. This means that the analysis of table (15) is exactly that of Section 4, except that $\mathbf{w}_{1-\lambda}$ assumes the role of $\mathbf{w}_\lambda$; for example, the antiplurality and plurality methods swap roles, properties, illustrating examples, and peculiarities. This is, of course, a special case of equation (14). For instance, the antiplurality ($\lambda = 1$) outcome is $C \succ B \succ A$ for $(x, y) = (\frac{6}{15}, \frac{5}{15})$ from FIGURE 7a. As this profile is the reversal of the beverage example equation (1) with plurality ($\lambda = 0$) outcome $A \succ B \succ C$, the outcome is as Theorem 4 requires.

An easy way to use Theorem 4 to convert results from Section 4 to the current setting is to add or subtract 3 from all of the type numbers of FIGURE 5 and FIGURE 6, and replace statements about λ with statements about $1 - \lambda$. This completes the analysis for the reversed beverage examples. It means, for instance, that only nine voters are needed to create an example where all candidates can be elected with some $\mathbf{w}_\lambda$ and that the likelihood of this occurring is higher than the likelihood of each candidate being bottom-ranked by some procedure. Namely, the reversal of preferences reverses the conclusions obtained from FIGURE 6. Only the Borda Count has essentially identical conclusions for both settings; this is because $\lambda = \frac{1}{2}$ is the only procedure allowing $\mathbf{w}_\lambda = \mathbf{w}_{1-\lambda}$. Incidentally, this symmetry condition turns out to be a technical reason which ensures that the Borda Count has strongly favorable properties.

By applying this analysis along with equation (14) to all of the settings in table (13), we account for six more settings. This leaves only six more to consider.

Final case The final situation is where voters come from types 1, 2, and 3. There are no real surprises in the analysis, so it is left for the interested reader. By use of the symmetry of neutrality, the same analysis extends to the six remaining cases.

6. Summary

Surprisingly subtle, unexpected election behaviors can arise when voters are restricted to only three kinds of preferences. Of particular interest is that the questions raised in Section 1 about potential paradoxical election behavior can be answered by using elementary geometric arguments. As shown, conflict between pairwise and positional methods occurs in abundance and, when it occurs, it is supported by an open set of profiles. (This answers the robustness question.) Problems about the likelihood of strange behavior, or finding supporting profiles with the minimum number of voters, reduce to elementary arguments. Moreover, the geometry allows us to "see" where conflict occurs and to determine whether paradoxical outcomes are, or are not, isolated. For instance, FIGURE 6 identifies the profiles where each candidate wins with an appropriate $\mathbf{w}_\lambda$ method. So, when preferences are restricted as indicated, we must expect such pathological behavior in about 1 in 40 elections (with a sufficient number of voters). As shown by FIGURE 7, other settings increase the likelihood of this behavior to about 3 in 20 elections.

Although we emphasized those election surprises that occur when voters' preferences come from only three possible types, other surprises already occur when preferences are restricted to only two types. Indeed, this is a special case of our analysis because it just requires setting one of x, y, or z equal to zero; it is the behavior on one of the edges of the triangles T_1, T_2, or T_3. For instance, by considering the vertical leg (where $x = 0$) of the triangles in FIGURE 5, we discover how this highly restrictive case allows two strict pairwise rankings to be accompanied with conflicting $\mathbf{w}_\lambda$ outcomes. Without question, elections admit surprising behavior.

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