ARTICLES

Permutations and Combination Locks

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Consider a combination lock with n buttons, numbered 1 through n. A valid combination consists of a sequence of button-pushes, in which each button is pushed exactly once. If the buttons must be pushed one at a time, then clearly there will be $n!$ possible combinations. But what if we are allowed to push buttons simultaneously?

We can represent a valid combination for such a lock as a sequence of disjoint, nonempty subsets of the set $B = \{1, 2, \dots, n\}$ whose union is B. Each set in the sequence specifies a collection of buttons to be pushed simultaneously. For example, if $n = 3$ then we have the following possible combinations:

$$(\{1\},\{2\},\{3\}),(\{1\},\{3\},\{2\}),(\{2\},\{1\},\{3\}),(\{2\},\{3\},\{1\}),$$
$$(\{3\},\{1\},\{2\}),(\{3\},\{2\},\{1\}),(\{1,2\},\{3\}),(\{1,3\},\{2\}),(\{2,3\},\{1\}),$$
$$(\{1\},\{2,3\}),(\{2\},\{1,3\}),(\{3\},\{1,2\}),(\{1,2,3\}).$$

In this paper we will study formulas for the number of valid combinations for such a lock.

One of the most captivating features of this problem is that it may be solved by a number of elementary methods. In particular, we will use recurrence relations and generating functions to derive a variety of solutions ranging from the values of certain derivatives, to an infinite series, to some double summations with natural combinatorial interpretations. To obtain an asymptotic formula, we will use a basic integral estimate for our infinite series solution. It was more than a coincidence that, while we were developing these results, both of us were teaching second-semester calculus. Indeed we have endeavored to keep our calculus students in mind as we were writing.

A second striking feature of the lock combination problem is that it leads naturally to some well-known integer sequences. The reason we have included "Permutations" in the title is revealed by the appearance in Section 2 of the Eulerian numbers, which count the cardinalities in a natural partitioning of the set of permutations of n objects. We will also observe a connection between our lock problem and the Stirling numbers of the second kind.

We note that a number of authors have studied similar problems (Cayley **[3]**, Good **[7]**, and Gross **[8]**), though it appears that only Borenius, Danielson, and Jansson **[1]** have observed the connection with the Eulerian numbers. While most of our formulas can be found in these sources, we hope that readers will share our delight in collecting them and providing them with elementary derivations. We would like to thank Stan Wagon for sparking our interest in this problem and informing us of a commercially available door lock of this type with five buttons.

1. Two Solutions and an Asymptotic Formula

Let a_n be the number of combinations for a lock with n buttons. If $n = 0$ then the only valid combination is the empty sequence, so $a_0 = 1$. For $n > 0$, a valid combination will consist of a collection of k buttons that are pushed simultaneously, for some $1 \le k \le n$, followed by a combination using the remaining $n - k$ buttons. Thus we are led to the following recurrence relation for a_n:

$$a_0 = 1, \qquad a_n = \binom{n}{1}a_{n-1} + \binom{n}{2}a_{n-2} + \cdots + \binom{n}{n}a_0 \quad \text{for } n > 0.$$

Using this recurrence, we can compute the following values of a_n, for $n \le 10$:

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	1	3	13	75	541	4683	47293	545835	7087261	102247563

Note that the list of combinations above for the case $n = 3$ confirms the value $a_3 = 13$ in this table.

Filling in the formula for the binomial coefficients in the recurrence above, we find a common factor of $n!$:

$$a_n = n!\left(\frac{a_{n-1}}{1!(n-1)!} + \frac{a_{n-2}}{2!(n-2)!} + \cdots + \frac{a_0}{n!0!}\right).$$

This suggests defining $b_n = a_n/n!$. Dividing the formula above by $n!$ yields the following recurrence relation for b_n:

$$b_0 = 1, \qquad b_n = b_{n-1} + \frac{b_{n-2}}{2!} + \cdots + \frac{b_0}{n!} \quad \text{for } n > 0.$$

This recurrence relation is slightly simpler than the one for a_n, which suggests that we might be able to find a formula for a_n by first solving the recurrence for b_n. We begin with some simple bounds on b_n.

THEOREM 1. *For all n,*

$$\frac{1}{2(\ln 2)^n} \le b_n \le \frac{1}{(\ln 2)^n}.$$

Proof. We proceed by induction on n. The inequalities in the theorem clearly hold when $n = 0$. Now suppose $n > 0$, and assume that the inequalities hold for all $n' < n$. Then

$$\begin{aligned} b_n &= b_{n-1} + \frac{b_{n-2}}{2!} + \cdots + \frac{b_0}{n!} \\ &\le \frac{1}{(\ln 2)^{n-1}} + \frac{1}{2!(\ln 2)^{n-2}} + \cdots + \frac{1}{n!} \\ &= \frac{1}{(\ln 2)^n}\left(\ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!}\right) \\ &\le \frac{1}{(\ln 2)^n}(e^{\ln 2} - 1) = \frac{1}{(\ln 2)^n}. \end{aligned}$$

This gives the desired upper bound on b_n. For the lower bound, we begin by applying the induction hypothesis to all terms in the formula for b_n *except* the last:

$$\begin{aligned} b_n &= b_{n-1} + \frac{b_{n-2}}{2!} + \cdots + \frac{b_1}{(n-1)!} + \frac{b_0}{n!} \\ &\geq \frac{1}{2(\ln 2)^{n-1}} + \frac{1}{2!\cdot 2(\ln 2)^{n-2}} + \cdots + \frac{1}{(n-1)!\cdot 2\ln 2} + \frac{1}{n!} \\ &= \frac{1}{2(\ln 2)^n}\left(\ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^{n-1}}{(n-1)!} + \frac{2(\ln 2)^n}{n!}\right). \end{aligned}$$

By Taylor's Theorem, there is a number c such that $0 < c < \ln 2$ and

$$\begin{aligned} e^{\ln 2} &= 1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^{n-1}}{(n-1)!} + \frac{e^c(\ln 2)^n}{n!} \\ &\leq 1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^{n-1}}{(n-1)!} + \frac{2(\ln 2)^n}{n!}. \end{aligned}$$

Thus

$$\begin{aligned} b_n &\geq \frac{1}{2(\ln 2)^n}\left(\ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^{n-1}}{(n-1)!} + \frac{2(\ln 2)^n}{n!}\right) \\ &\geq \frac{1}{2(\ln 2)^n}(e^{\ln 2} - 1) = \frac{1}{2(\ln 2)^n}, \end{aligned}$$

as required.

A natural way to study the sequence $\{b_n\}$ is to define the generating function

$$f(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Note that by Theorem 1, the sum converges absolutely for $|x| < \ln 2$. Using our recurrence relation for b_n, we can solve for $f(x)$:

$$\begin{aligned} f(x) &= b_0 + \sum_{n=1}^{\infty} b_n x^n = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{b_{n-k}}{k!}x^n = 1 + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{b_{n-k}}{k!}x^n \\ &= 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}\sum_{n=k}^{\infty} b_{n-k}x^{n-k} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}\sum_{n=0}^{\infty} b_n x^n = 1 + (e^x - 1)f(x), \end{aligned}$$

and therefore

$$f(x) = \frac{1}{2 - e^x}, \qquad |x| < \ln 2.$$

Since b_n is the coefficient of x^n in the Maclaurin series for $f(x)$, we have $b_n = f^{(n)}(0)/n!$. But recall that $a_n = n! b_n$, so we have proven the following theorem:

THEOREM 2. *For all* n,

$$a_n = \frac{d^n}{dx^n}\left(\frac{1}{2 - e^x}\right)\bigg|_{x=0}.$$

To compute values of a_n using Theorem 2, we must compute derivatives of the function $1/(2 - e^x)$. One way to make this computation easier is to rewrite the function as a geometric series:

$$\frac{1}{2 - e^x} = \frac{1/2}{1 - e^x/2} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{e^x}{2}\right)^k, \qquad x < \ln 2.$$

Differentiating term-by-term we find that

$$\frac{d^n}{dx^n}\left(\frac{1}{2-e^x}\right)=\frac{1}{2}\sum_{k=0}^{\infty}k^n\left(\frac{e^x}{2}\right)^k, \qquad x<\ln 2.$$

(Note that for $n=0$ we use the convention $0^0=1$.) Thus, applying Theorem 2 we get another formula for a_n:

THEOREM 3. *For all* n,

$$a_n=\frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k}.$$

Using Theorem 3, we can improve on our estimates in Theorem 1. A natural way to estimate the infinite sum in Theorem 3 is to use the improper integral

$$\int_0^{\infty}\frac{x^n}{2^x}dx.$$

We evaluate this integral by performing the substitution $u=x\ln 2$ and then integrating by parts repeatedly (or recognizing the Γ function):

$$\int_0^{\infty}\frac{x^n}{2^x}dx=\frac{1}{(\ln 2)^{n+1}}\int_0^{\infty}\frac{u^n}{e^u}du=\frac{\Gamma(n+1)}{(\ln 2)^{n+1}}=\frac{n!}{(\ln 2)^{n+1}}.$$

Thus, by Theorem 3 we expect to have

$$a_n\approx\frac{n!}{2(\ln 2)^{n+1}}.$$

For small n, this approximation is remarkably accurate. For example, we have:

$$a_{15}=230283190977853,$$
$$\frac{15!}{2(\ln 2)^{16}}\approx 230283190977853.04.$$

Unfortunately, for larger n the error of the approximation grows, but we can place bounds on this error.

THEOREM 4. *For all* n,

$$\frac{n!}{2(\ln 2)^{n+1}}-\frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n<a_n<\frac{n!}{2(\ln 2)^{n+1}}+\frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n.$$

Proof. It is easy to check that the function $g(x)=x^n/2^x$ is increasing on the interval $[0,n/\ln 2]$ and decreasing on $[n/\ln 2,\infty)$, and thus its maximum value on the interval $[0,\infty)$ is

$$g(n/\ln 2)=\frac{(n/\ln 2)^n}{2^{n/\ln 2}}=\left(\frac{n}{e\ln 2}\right)^n.$$

Let j be the largest integer less than or equal to $n/\ln 2$ and put $r=(n/\ln 2)-j$. Thus $(n/\ln 2)=j+r$, j is a nonnegative integer, and $0\le r<1$.

We now use upper and lower rectangles to overestimate and underestimate the integral of $g(x)$. Upper rectangles of width 1 lead to the following overestimate of the integral (see FIGURE 1):

$$\int_0^\infty \frac{x^n}{2^x}dx < \sum_{k=1}^{j}\frac{k^n}{2^k} + \left(\frac{n}{e\ln 2}\right)^n + \sum_{k=j+1}^{\infty}\frac{k^n}{2^k} \le 2a_n + \left(\frac{n}{e\ln 2}\right)^n.$$

Substituting in the value of the integral gives the required lower bound on a_n.

To underestimate with lower rectangles we again use rectangles of width 1, except that we split the interval from j to $j+1$ into two rectangles, of width r and $1-r$ (see FIGURE 2). This gives us

$$\begin{aligned}\int_0^\infty \frac{x^n}{2^x}dx &> \sum_{k=0}^{j-1}\frac{k^n}{2^k} + r\frac{j^n}{2^j} + (1-r)\frac{(j+1)^n}{2^{j+1}} + \sum_{k=j+2}^{\infty}\frac{k^n}{2^k}\\ &= 2a_n - \left((1-r)\frac{j^n}{2^j} + r\frac{(j+1)^n}{2^{j+1}}\right)\\ &\le 2a_n - \left(\frac{n}{e\ln 2}\right)^n,\end{aligned}$$

which leads to the stated upper bound on a_n.

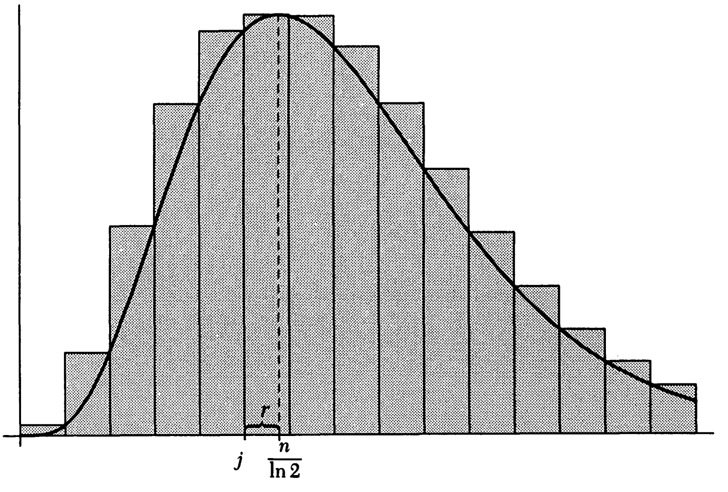

FIGURE 1

Approximating $\int_0^\infty \frac{x^n}{2^x}\,dx$ with upper rectangles.

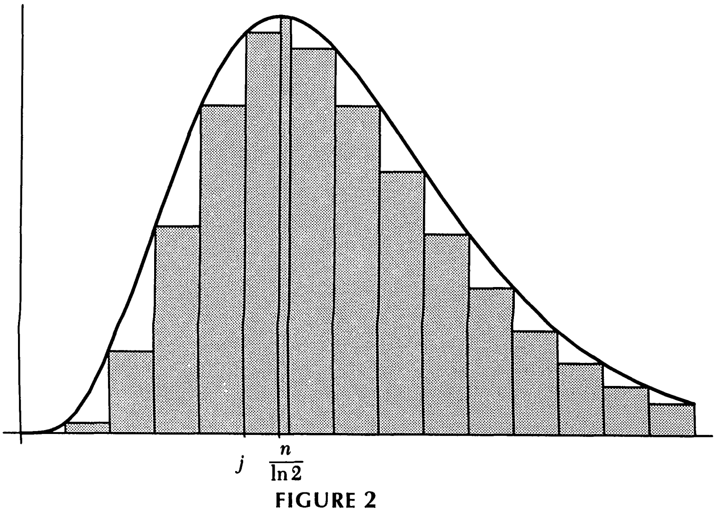

FIGURE 2

Approximating $\int_0^\infty \frac{x^n}{2^x}\,dx$ with lower rectangles.

Better bounds on a_n can be obtained by using methods from complex analysis (see **[1]**, **[7]**, and **[8]**). However, our bounds are good enough to prove:

COROLLARY 5.

$$\lim_{n\to\infty} \frac{a_n}{n!/\left[2(\ln 2)^{n+1}\right]} = 1.$$

Proof. By Theorem 4, we have

$$\left| \frac{a_n}{n!/\left[2(\ln 2)^{n+1}\right]} - 1 \right| < \frac{\ln 2 (n/e)^n}{n!}.$$

To complete the proof of the corollary, we show that the quantity on the right side of this inequality approaches 0 as n approaches infinity. Recall Stirling's formula, which says that

$$\lim_{n\to\infty} \frac{\sqrt{2\pi n}\,(n/e)^n}{n!} = 1.$$

Applying Stirling's formula, we find that

$$\lim_{n\to\infty} \frac{\ln 2(n/e)^n}{n!} = \lim_{n\to\infty} \frac{\ln 2}{\sqrt{2\pi n}} \frac{\sqrt{2\pi n}\,(n/e)^n}{n!} = 0\cdot 1 = 0.$$

2. A Connection with Permutations

So far we have been concentrating on estimates of the sum in Theorem 3, in order to get approximations to a_n. We can also evaluate the sum exactly, as follows. For $n \geq 0$ let

$$h_n(x) = \sum_{k=0}^{\infty} k^n x^k.$$

Then by Theorem 3, $a_n = (1/2)h_n(1/2)$. We now derive a formula for a_n by finding formulas for the $h_n(x)$'s.

Clearly

$$h_0(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \qquad -1 < x < 1.$$

Also, differentiating term-by-term we see that

$$h_n'(x) = \sum_{k=0}^{\infty} k^{n+1} x^{k-1}, \qquad \text{so} \quad x\,h_n'(x) = \sum_{k=0}^{\infty} k^{n+1} x^k = h_{n+1}(x).$$

Applying this recurrence repeatedly, we get the following formulas for $h_n(x)$, for $n \leq 5$:

$$h_0(x) = \frac{1}{1-x}, \qquad h_1(x) = \frac{x}{(1-x)^2},$$

$$h_2(x) = \frac{x+x^2}{(1-x)^3}, \qquad h_3(x) = \frac{x+4x^2+x^3}{(1-x)^4},$$

$$h_4(x) = \frac{x+11x^2+11x^3+x^4}{(1-x)^5}, \qquad h_5(x) = \frac{x+26x^2+66x^3+26x^4+x^5}{(1-x)^6}.$$

It appears that $h_n(x)$ is always a polynomial of degree n divided by $(1-x)^{n+1}$, but it is not immediately clear what the pattern of coefficients in these polynomials is. To study this pattern, we will introduce the following notation. Let $A_{n,k}$ be the coefficient of x^k in the numerator of $h_n(x)$. For example, $A_{5,4}=26$. Then the recurrence for $h_n(x)$ above can be used to derive a recurrence for the $A_{n,k}$'s:

THEOREM 6. *For all $n \geq 1$,*

$$h_n(x) = \frac{\sum_{k=1}^{n} A_{n,k} x^k}{(1-x)^{n+1}},$$

where the numbers $A_{n,k}$ are given by the following recurrence relation:

$$A_{n,1} = A_{n,n} = 1, \qquad A_{n+1,k} = kA_{n,k} + (n+2-k)A_{n,k-1} \quad \textit{for } 2 \leq k \leq n.$$

Proof. We proceed by induction. The case $n=1$ is easy to verify. For the induction step, suppose the formula in the statement of the theorem is correct for h_n. Then

$$\begin{aligned}
h_{n+1}(x) &= xh_n'(x) = x\frac{d}{dx}\left(\frac{\sum_{k=1}^{n} A_{n,k} x^k}{(1-x)^{n+1}}\right) \\
&= x\frac{(1-x)^{n+1}\sum_{k=1}^{n} kA_{n,k}x^{k-1} + (n+1)(1-x)^n \sum_{k=1}^{n} A_{n,k}x^k}{(1-x)^{2n+2}} \\
&= \frac{\sum_{k=1}^{n} kA_{n,k}(1-x)x^k + \sum_{k=1}^{n}(n+1)A_{n,k}x^{k+1}}{(1-x)^{n+2}} \\
&= \frac{\sum_{k=1}^{n} kA_{n,k}x^k + \sum_{k=1}^{n}(n+1-k)A_{n,k}x^{k+1}}{(1-x)^{n+2}} \\
&= \frac{x + \sum_{k=2}^{n}\left(kA_{n,k} + (n+2-k)A_{n,k-1}\right)x^k + x^{n+1}}{(1-x)^{n+2}} \\
&= \frac{\sum_{k=1}^{n+1} A_{n+1,k}x^k}{(1-x)^{n+2}}.
\end{aligned}$$

The recurrence relation given in Theorem 6 allows us to readily compute the coefficients $A_{n,k}$ for $1 \leq k \leq n$, given the coefficients $A_{n-1,k}$ for $1 \leq k \leq n-1$. Thus it is convenient to display these numbers in the form of a triangular table à la Pascal's triangle. The nth row of the triangle will be

$$A_{n,1} \quad A_{n,2} \quad \cdots \quad A_{n,n}.$$

These rows may be computed by beginning with $A_{1,1}=1$ and generating each row from the preceding row via the recurrence relation of Theorem 6. The first seven rows of the triangle are shown in FIGURE 3.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 1 & & & & \\
 & & & 1 & & 11 & & 11 & & 1 & & & \\
 & & 1 & & 26 & & 66 & & 26 & & 1 & & \\
 & 1 & & 57 & & 302 & & 302 & & 57 & & 1 & \\
1 & & 120 & & 1191 & & 2416 & & 1191 & & 120 & & 1
\end{array}$$

FIGURE 3

Note that the first five rows agree with our computations of $h_n(x)$ above. Like Pascal's triangle, this triangle appears to be symmetric, which suggests that perhaps $A_{n,k} = A_{n,n+1-k}$ for all $1 \le k \le n$. An even more striking analogy with Pascal's triangle is revealed by adding the entries across the rows of our triangle. Recall that row n of Pascal's triangle adds up to 2^n, because the entries in this row count the numbers of subsets of $\{1,2,\dots,n\}$ of different sizes. Adding the entries across the rows of the triangle in FIGURE 3, we find that the sums of the rows are: $1, 2, 6, 24, 120, 720, 5040$; i.e., the factorials! This observation suggests that we look for an interpretation of the numbers $A_{n,k}$, for $1 \le k \le n$, in terms of a partition of the set of permutations of $\{1,2,\dots,n\}$.

For ease of terminology, we will use the term *n-permutation* to denote a permutation of $\{1,2,\dots,n\}$. We think of an n-permutation as an ordered list of the numbers $1,2,\dots,n$. Given a permutation in this form, say $s_1\, s_2 \dots s_n$, we count the number of increasing runs in the sequence $s_1, s_2, \dots, s_n$ reading from left to right. For example, the permutation 1 4 2 5 3 6 8 7 has four increasing runs; namely, 1 4, 2 5, 3 6 8, and 7. As the next proposition shows, partitioning the set of n-permutations according to their numbers of increasing runs yields the numbers $A_{n,k}$.

PROPOSITION 7. *For all n and k such that* $1 \le k \le n$, *the number of n-permutations with k increasing runs is* $A_{n,k}$.

Proof. See [**4**, Theorem A, pp. 241–2] or [**16**, Problem 12.22(a)].

The numbers $A_{n,k}$, and the interpretation given for them in Proposition 7, are well known in combinatorics (see [**4**], [**13**], [**14**], [**15**], and [**16**]). They are called the *Eulerian numbers* (see [**5**]), and so it is appropriate to refer to the triangular table of numbers we defined above as *Euler's triangle*. MacMahon [**11**] was apparently the first to use Eulerian numbers to classify permutations by increasing runs. The Eulerian numbers have appeared in a variety of contexts in combinatorics and statistics (e.g., see [**2**], [**9**], and [**10**]). A very extensive description of their theory is given by Foata and Schützenberg [**6**].

We can now confirm the conjectures we made based on the first seven rows of Euler's triangle.

COROLLARY 8. *For all* $n \ge 1$,

(a) $\sum_{k=1}^{n} A_{n,k} = n!$ *and*

(b) $A_{n,k} = A_{n,n+1-k}$ *for* $1 \le k \le n$.

Proof. (a) is immediate from Proposition 7. For a proof of (b), see [**4**, Theorem B, p. 242] or [**16**, Problem 12.22(b)].

Now we return to our original lock problem. From Theorems 3 and 6, we see that the number of combinations for a lock with $n \ge 1$ buttons, a_n, satisfies:

$$a_n = \frac{1}{2} h_n\left(\frac{1}{2}\right) = \frac{1}{2} \sum_{k=1}^{n} A_{n,k} \left(\frac{1}{2}\right)^{k-n-1} = \sum_{k=1}^{n} A_{n,k} 2^{n-k}. \tag{1}$$

By the symmetry of Euler's triangle stated in Corollary 8(b), we also have:

$$a_n = \sum_{k=1}^{n} A_{n,n+1-k}\, 2^{(n+1-k)-1} = \sum_{k=1}^{n} A_{n,k} 2^{k-1}. \tag{2}$$

Equation (1) has a natural interpretation in terms of lock combinations. To see this, first note that given a lock combination, we can generate a corresponding n-permutation by writing down the numbers of the buttons pressed, in the order in which they are pressed, with the numbers of the buttons pressed simultaneously being written in increasing order. For example, the lock combination $(\{1\},\{3,4\},\{2,5\})$ would correspond to the permutation 1 3 4 2 5. Clearly a lock combination that consists of l steps will lead to a permutation with at most l increasing runs. Furthermore, if we follow this procedure for every lock combination, each permutation with k increasing runs will appear exactly 2^{n-k} times. To demonstrate this, let σ be an n-permutation with k increasing runs, and consider separating this permutation into blocks representing the steps in a corresponding lock combination; for example, we might do this by inserting vertical lines in the permutation to delimit the blocks. The numbers within each block must be in increasing order, so we must at least insert the $k-1$ vertical lines needed to delimit the k increasing runs in σ. There are $n-1$ spaces between the n numbers where vertical lines might go, and $k-1$ of them are now filled, so there are $(n-1)-(k-1)=n-k$ places left unfilled. By choosing a subset of these $n-k$ positions and inserting vertical lines at the chosen positions, we determine a lock combination corresponding to σ. Hence, there are 2^{n-k} lock combinations corresponding to a single n-permutation with k increasing runs.

For example, the permutation 1 3 4 2 5 has two increasing runs: 1 3 4 and 2 5. It corresponds to the $2^{5-2}=8$ lock combinations:

$$(\{1,3,4\},\{2,5\}),(\{1\},\{3,4\},\{2,5\}),(\{1,3\},\{4\},\{2,5\}),(\{1\},\{3\},\{4\},\{2,5\}),$$
$$(\{1,3,4\},\{2\},\{5\}),(\{1\},\{3,4\},\{2\},\{5\}),$$
$$(\{1,3\},\{4\},\{2\},\{5\}),(\{1\},\{3\},\{4\},\{2\},\{5\}).$$

Proceeding by analogy with Pascal's triangle and, in particular, in view of the recurrence relation of Theorem 6, one might hope for an explicit formula giving the numbers $A_{n,k}$ in terms of binomial coefficients. In fact, such a formula is known.

THEOREM 9. *For* $1 \le k \le n$,

$$A_{n,k}=\sum_{i=0}^{k-1}(-1)^i\binom{n+1}{i}(k-i)^n.$$

Proof. See [**4**, Theorem C, p. 243], [**16**, Problem 12.22(d)], or [**12**, Theorem 2].

Combining equations (1) and (2) with Theorem 9, we obtain two more formulas for a_n.

THEOREM 10. *For all* $n \ge 1$,

$$a_n=\sum_{k=1}^{n}\sum_{i=0}^{k-1}(-1)^i\binom{n+1}{i}(k-i)^n 2^{n-k}=\sum_{k=1}^{n}\sum_{i=0}^{k-1}(-1)^i\binom{n+1}{i}(k-i)^n 2^{k-1}.$$

Our last solution to the combination lock problem involves another well-known family of numbers, the *Stirling numbers of the second kind*. For $1 \le k \le n$, the Stirling number $S(n,k)$ is the number of unordered partitions of the set $\{1,2,\dots,n\}$ into k nonempty subsets. Since a lock combination with k steps can be thought of as an *ordered* partition of this set into k nonempty subsets, we see that the number of lock combinations with k steps is equal to $k!S(n,k)$. Summing over all k gives us the following formula:

THEOREM 11. *For all* $n \geq 1$,

$$a_n = \sum_{k=1}^{n} k!S(n,k).$$

Like the Eulerian numbers, the Stirling numbers of the second kind can also be expressed in terms of binomial coefficients. In fact, there is a striking resemblance between the formula for $S(n,k)$ given in our next theorem and the formula for $A_{n,k}$ given in Theorem 9.

THEOREM 12. *For all* $1 \leq k \leq n$,

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n.$$

Proof. See [**4**, Theorem A, pp. 204–205].

Combining Theorems 11 and 12, we obtain our last formula for a_n.

THEOREM 13. *For all* $n \geq 1$,

$$a_n = \sum_{k=1}^{n} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n.$$

In conclusion, we note that we can now compute the number of valid combinations for the commercially available five-button combination door lock mentioned in the introduction. This lock has an extra feature; namely, it does not require that all five buttons be used in a valid combination. For example, $(\{2\})$, $(\{1,5\},\{3\})$, and $(\{1,2,4,5\})$ are valid combinations. By viewing the subset of unused buttons as the last block in a combination that uses every button, we see that there is a one-to-one correspondence between the combinations that use all the buttons and those that don't. However, this correspondence includes the empty combination—the combination in which no buttons are pushed—and this combination is not meaningful for a commercial door lock. Thus, the number of meaningful combinations for the lock is $2a_5 - 1 = 1081$.

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A Mnemonic For e

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Mnemonic devices for the decimal expansion of π are well known. For example, "May I have a large container of coffee" may be found in [**1**]. The principle is to replace each word by the number of letters in it. Similar mnemonics for π exist in other languages, including Russian and Greek.

On the other hand, the only mnemonics for e known to the author are those given in [**2**] (in English, French, and Spanish) covering 10 places after the decimal point and [**3**] containing a sentence going to 20 digits. Potential authors may have been discouraged by early appearances of zeros in

$$e = \mathbf{2.718281828459045235360287471352662497757}\ldots.$$

The 20-place mnemonic in [**3**] uses the (one-letter) word "O" to represent the zero at place 13.

In the quasi-poem below, zeros are represented by exclamations "...!" whose canonical pronunciation is a short "ah!" (although the readers may substitute their favorite letterless words).

We present a mnemonic
    To memorize a constant
So exciting that Euler exclaimed: "...!"
    When first it was found,
Yes, loudly: "...!"
    My students perhaps
Will compute e,
    Use power of Taylor series,
An easy summation formula,
    Obvious, clear, elegant!

The author thanks the referee for making the author aware of [**2**] and [**3**] and for suggesting an improvement.

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