# The Geometry of Harmonic Functions

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## 1. Introduction

Imagine a society in which the citizens are encouraged, indeed compelled up to a certain age, to read (and sometimes write) musical scores. All quite admirable. However, this society also has a very curious (few remember how it all started) and disturbing law: *Music must never be listened to or performed!*

Though its importance is universally acknowledged, for some reason music is not widely appreciated in this society. To be sure, professors still excitedly pore over the great works of Bach, Wagner, and the rest, and they do their utmost to communicate to their students the beautiful meaning of what they find there, but they still become tongue-tied when brashly asked the question, "What's the point of all this?!"

In this parable, it was patently unfair and irrational to have a law forbidding would-be music students from experiencing and understanding the subject directly through "sonic intuition." But in our society of mathematicians we *have* such a law. It is not a written law, and those who flout it may yet prosper, but it says, *Mathematics must not be visualized!*

More likely than not, when a mathematics student today opens a random text on a random subject, he is confronted by abstract symbolic reasoning that is divorced from his sensory experience of the world, *despite* the fact that the very phenomena he is studying were often discovered by appealing to geometric (and perhaps physical) intuition. This reflects the fact that steadily over the last hundred years the honor of visual reasoning in mathematics has been besmirched. Only recently have many mathematicians picked up the gauntlet on its behalf, openly challenging the current dominance of purely symbolic logical reasoning.

Rather than indulge in further pulpit-thumping, we refer the sympathetic reader to a cheering MAA book [**1**]. The present author has joined the fray by attempting to render palpable the beautiful truths of elementary complex analysis by means of new geometric insights. Both this paper and a previous one [**2**] arose in connection with that work [**3**].

Our concern here will be with the various formulae for expressing a harmonic function in the interior of a planar region in terms of its values on the boundary. In place of the usual symbolic arguments, we shall supply simple geometric explanations/derivations of these formulae. But before we begin to visualize these formulae (starting in the next section) we must clarify the nature of the questions to which they are the answers. We begin with Poisson's formula for the disk.

Think of the complex plane as a thermally insulated sheet of metal; heat flows freely within it, but does not leak away into the surrounding space. Now supply heat at a constant rate to various points (*sources*) of the plane, and likewise remove heat at other places (*sinks*). Initially, the temperature of the metal at any given point will vary with time. A small element of the metal plate gains or gives up energy as heat attempts to flow across it from the sources to the sinks. But eventually (quickly, if the thermal conductivity is high) the heat flow will settle down into a steady pattern and the temperature at a point $z$ will likewise settle to a definite value $T(z)$. In this steady state, the global statement of the conservation of energy is that total heat supplied at the sources equals the total heat removed at the sinks (possibly including infinity).

It is shown in elementary physics that $-\nabla T$ is the heat flow vector, and it then follows that the *local* statement of the indestructibility of energy is that, in the steady state, $T(z)$ is *harmonic*: Away from sources and sinks, it satisfies Laplace's equation,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \nabla \cdot (\nabla T)$$
$$= -(\textit{local rate of energy production per unit area})$$
$$= 0.$$

Suppose that we now measure the temperature around the circumference $C$ of a circle of radius $R$, the interior of which is free of sources and sinks, and the center of which we conveniently choose to be the origin. We hope that it may seem physically plausible that these values actually determine the temperature at any interior point $a$. This was confirmed by Poisson in 1820 when he derived an explicit formula for $T(a)$ in terms of $T(C)$.

As $z = Re^{i\theta}$ moves around $C$, we may express the measured temperature as a function of the angle: $T = T(\theta)$. *Poisson's formula* is then

$$T(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{R^2 - |a|^2}{|z-a|^2}\right]T(\theta)\,d\theta. \tag{1}$$

The quantity in square brackets is called the *Poisson kernel*, and we shall write it $\mathscr{P}_a(z)$. Thus (1) may be roughly paraphrased as saying that the heat of an element of $C$ at $z$ propagates to $a$ with a "facility" $\mathscr{P}_a(z)$ that dies away as the square of the distance between $z$ and $a$.

This formula is connected with an important and difficult issue that engaged an illustrious cast of characters: Riemann, Weierstrass, Schwarz, Klein, Poincaré, and Hilbert. Instead of dealing with a pre-existing harmonic function, *Dirichlet's problem* demands that we arbitrarily (but piecewise continuously) assign values to the boundary of a simply connected region $R$ and then inquire if there always exists a harmonic function in $R$ that takes on these values as the boundary is approached. (This problem is in fact closely related to the equally famous *Plateau's problem*: Given a simple closed curve in space, does there exist a minimal surface that spans it?)

In the case of the disk, H. A. Schwarz demonstrated that not only does the solution to Dirichlet's problem exist, but it is explicitly given by (1). If we are handed the piecewise-continuous values $T(\theta)$ on $C$ then we may construct a function $T(a)$ in the interior according to Poisson's recipe. Schwarz's solution then amounted to showing that $T(a)$ is automatically harmonic, and that as $a$ approaches a boundary point at which $T(\theta)$ is continuous, $T(a)$ approaches the given value $T(\theta)$. The truth of all this will be explained in the next two sections.

If we assume (as was implicit in the previous discussion) that mathematical harmonic functions are *identical* with physical temperature distributions, then both the existence and uniqueness of a mathematical solution to Dirichlet's problem for general regions is assured by Nature's solution to the equivalent physical problem: Heat the boundary points of $R$ to their assigned temperatures; let things settle down; the temperature in the interior is then the desired harmonic function. In like manner, if we assume the identity between soap films and minimal surfaces, we need only dip a bent loop of wire into soapy water in order to solve Plateau's problem.

A sliver of history: Riemann *did* make the above identification (actually, he thought in terms of electricity rather than heat) and thereby reaped a rich harvest of mathematical discoveries based on physical intuition; in particular, later we shall see how it led him to his mapping theorem. Perhaps aware of the audacity of his style of reasoning, Riemann sought to bolster his physical intuition with a more mathematical idea. In ignorance of its earlier use by Gauss and Lord Kelvin, he christened this idea *Dirichlet's Principle*. Roughly, it asserts that if we consider the functions $T(a)$, continuous in the interior of $R$ and taking on prescribed boundary values, then, due to the nonnegative integrand, there *must be* one (don't you think?) that minimizes

$$\int\int_R \left[ \left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial y} \right)^2 \right] dx\,dy.$$

But it can then be shown that this minimizing function is also the solution to Dirichlet's problem, the existence of which was sought by Riemann.

However, in 1869 the brilliant but dryly logical Weierstrass threw a wrench into the works when he produced a *counterexample* to the general idea underlying Dirichlet's Principle. As Felix Klein later described the situation, "With this a large part of Riemann's developments came to nought." These clouds of doubt would continue to hang in the air for three decades. Undeterred, mathematicians such as Klein—who did not shun Weierstrassian rigor but was himself driven by geometric/physical intuition—continued to expound and extend Riemann's ideas. Only in 1900 did Hilbert finally "resurrect Dirichlet's Principle" (to use his own words) by showing that although Weierstrass had discredited the general idea behind it, this particular instance actually *is* correct.

In fairness, it should be explained that while Weierstrass doubted Riemann's proofs, he believed the results. Indeed, it was at his urging that Schwarz (a former pupil) found the above solution for the disk that did not rely on the suspect principle. Schwarz was then able to use this to show that a solution will also exist if $R$ is any *union* of disks.

It is clear to which camp the author owes his allegiance. Although Weierstrass would not approve, our mode of explanation will remain steadfastly geared to geometric intuition rather than logical rigor.

## 2. Schwarz's Interpretation

There is an exceedingly beautiful geometric interpretation of formula (1), due to Schwarz, which deserves to be far better known than it is. Of the myriad complex analysis texts, we have only found it described in the book by Ahlfors ([4], p.170). Schwarz obtained it, and likewise Ahlfors explains it, as a consequence of Poisson's formula (itself derived by computation). In this section we shall instead demonstrate Schwarz's result directly and geometrically, only then producing the Poisson formula as a consequence of *it*. First we remind the reader of some preliminary facts.

Suppose we measure the temperature at the center 0 of the circle $C$. If half of $C$ were at one temperature while the other half were at another, then symmetry and physical intuition would suggest that at 0 we would find the average of these temperatures. Dividing up $C$ further into arcs of constant temperature, then passing to the limit of infinitesimal arcs $R\,d\theta$ at temperatures $T(\theta)$, we are led to suspect *Gauss' mean value theorem* for harmonic functions:

$$T(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} T(\theta)\,d\theta. \tag{2}$$

This is often proved, as opposed to merely motivated, by appealing to Cauchy's Integral Formula; its consonance with the desired formula (1) is evident upon setting $a = 0$.

FIGURE 1(a) is intended to make the meaning of (2) more vivid. Imagine that there are thermometers placed all along the circle and that you are standing at 0. Turning your head successively through the same small angle marked • you would see the thermometers at the white dots on the boundary. The average of the temperatures you see (as • ⇝ 0) is then the temperature where you stand.

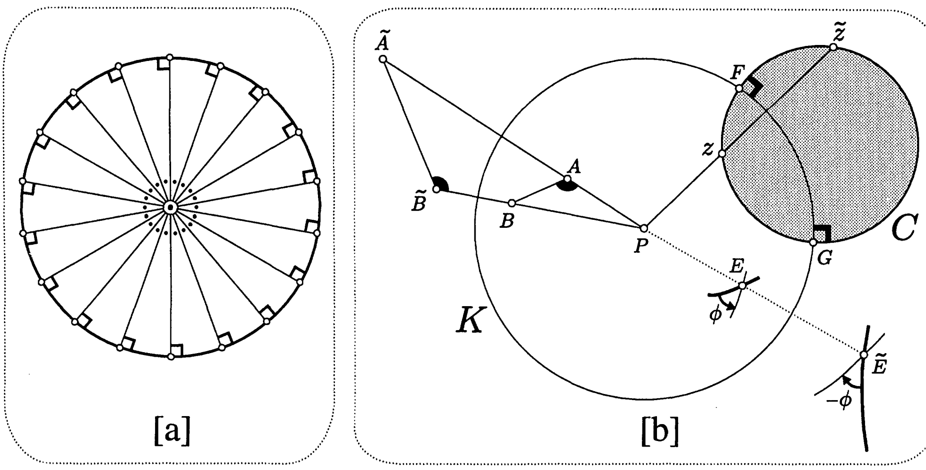


**FIGURE 1**

Next, recall that analytic mappings are essentially those that are *conformal*, that is, those for which the angle of intersection of two image curves is identical, in magnitude *and* sense, to that of the preimages. This makes it easy to understand that the composition $f[h(z)]$ of two analytic functions is itself analytic, for if two mappings preserve angles then so will their composition.

Since the real and imaginary parts of an analytic function $f(z) = T(z) + iS(z)$ are automatically harmonic by virtue of the Cauchy-Riemann equations, it follows from the above that $T[h(z)]$ is also harmonic. But if we are *given* a harmonic function $T(z)$ in a simply connected region, then it is always possible to find a *harmonic conjugate* $S(z)$ such that $f(z) = T(z) + i\, S(z)$ is analytic; in fact the level curves of $S$ are just the paths of the heat as it flows orthogonally across the isotherms $T = constant$. Hence, if $T(z)$ is any harmonic function and $h(z)$ any conformal mapping, then $T(z^*)$ is automatically harmonic, with $z^* = h(z)$.

Suppose now that $h(z)$ maps the disk to *itself*. If $z = Re^{i\theta}$ lies on $C$ then so does $z^* = Re^{i\theta^*}$, and, since we suppose that we have measured the temperature all around $C$, we therefore know the temperature $T(\theta^*)$ at $z^*$. Having the values of $T(\theta^*)$, we may now compute the integral in (2) for the harmonic function $T[h(z)]$ to obtain

$$T(0^*) = \frac{1}{2\pi}\int_{-\pi}^{\pi} T(\theta^*)\, d\theta, \tag{3}$$

in which it should be stressed that the averaging is still taking place with respect to the angle of $z$, not its image $z^*$.

We may interpret (3) as follows: To obtain the temperature at $0^*$ we need only take the average of the new temperature distribution on $C$ obtained by transplanting the temperature measured at each $z$ to the new location $z^*$. We are now halfway to Schwarz's result. To find the temperature at $a$ we must find a conformal mapping of the disk to itself such that 0 is sent to $a$, then take the average of the new temperature distribution. The chief surprise will be how simply the new distribution is related to the old one.

In order to find the mapping $h(z)$, we require the basic properties of *inversion* that are illustrated in FIGURE 1(b). Readers familiar with inversive geometry may readily skip to FIGURE 2.

Given a circle $K$ of radius $r$ and center $P$, recall that the point $\tilde{A}$ is the *inverse* of $A$ if it lies in the same direction from $P$ as $A$, and $PA \cdot P\tilde{A} = r^2$. If we consider a second point $B$ and its inverse $\tilde{B}$ then $(PA/PB) = (P\tilde{B}/P\tilde{A})$, and therefore the triangles $PAB$ and $P\tilde{B}\tilde{A}$ are similar. Thus,

$$\textit{the angles } PAB \textit{ and } P\tilde{B}\tilde{A} \textit{ are equal.} \tag{4}$$

It follows easily [exercise] from (4) that if two curves meet at angle $\phi$ in $E$ then their images under inversion in $K$ meet at angle $-\phi$ in $\tilde{E}$. In other words,

$$\textit{inversion is anticonformal.} \tag{5}$$

Before proceeding, remind yourself why it is that *circles map to circles*; this is again [exercise] an easy consequence of (4). Consider a disk such that its boundary circle $C$ cuts $K$ at right angles. Since $C$ is mapped to a circle and $F$ and $G$ remain fixed, it follows from (5) that

$$C \textit{ and its shaded interior are mapped onto themselves.} \tag{6}$$

In particular, the figure shows $z$ being mapped to $\tilde{z}$.

Returning to our original problem, the desired conformal mapping $h(z)$ is now within easy reach. See FIGURE 2. Through 0 and $a$ (at which the temperature is sought) draw the line $L$. Through $a$, draw the line perpendicular to $L$, meeting $C$ in $F$ and $G$ (not shown). Letting $P$ be the intersection of the tangents (not shown) at $F$ and $G$, draw the circle $K$ with center $P$ and radius $PF$. Since $K$ is orthogonal to $C$, (6) says the white disk is mapped to itself under inversion in $K$; furthermore, it is easy to see that the points 0 and $a$ are interchanged by the mapping. The only snag is that, by (5), the mapping is anticonformal rather than conformal. However, if we now *reflect in L* then the angle between two curves will be reversed a second time, thereby returning it to its original state. A viable conformal mapping is therefore

$$h(z) = \textit{inversion in } K\textit{, followed by reflection in } L. \tag{7}$$

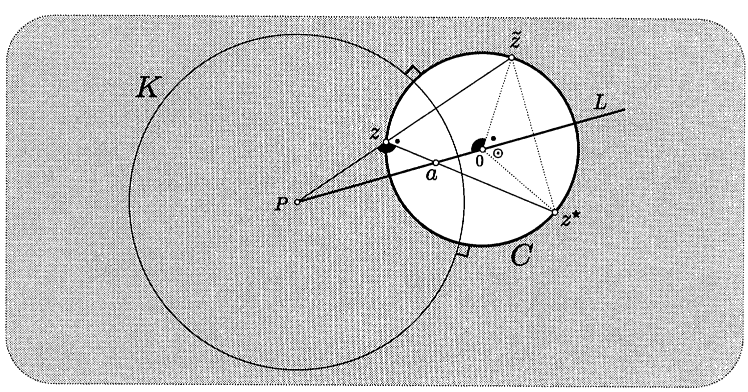


**FIGURE 2**

For readers with a smattering of hyperbolic geometry (which we will be using shortly) there is a simpler way of looking at $h$. [Excellent introductions to this geometry are [5] and [6].] The intersection point $m \equiv K \cap L$ of the two orthogonal hyperbolic lines $K$ and $L$ is the *midpoint* of the hyperbolic line segment $0a$. Inversion in $K$ corresponds to hyperbolic reflection in $K$, and, just as in Euclidean geometry, successively reflecting across two intersecting lines yields a rotation about their intersection point through double the angle contained by the lines. *Thus $h$ is a rotation of the hyperbolic plane through angle $\pi$ about the midpoint $m$*, making it easy to understand why the ends of the line segment are interchanged.

The geometric key to Schwarz's still unstated result lies in the following splendid fact. Instead of first sending $z$ to $\tilde{z}$ and then reflecting it to $z^*$, we may achieve the same thing in one fell swoop, and without needing $K$, by *projecting $z$ through $a$*. To see this, let us abuse our notation for a moment by defining $z^*$ to be this projected point; we must then show that it is the reflection of $\tilde{z}$ in $L$.

By (4), the similarly marked angles in FIGURE 2 are equal. But the angle subtended at 0 by $\tilde{z}$ and $z^*$ must be double that subtended on the circumference at $z$. The angles at 0 marked • and ⊙ must therefore be equal.

For a different approach, see [3].

We have thus bypassed all calculation and given a direct geometric demonstration of Schwarz's result: *To find the temperature at $a$, transplant each temperature on $C$ to the point directly opposite it as seen from $a$, then take the average of the new temperature distribution on $C$.*

The example in FIGURE 3 illustrates the beauty of this. In FIGURE 3(a), half of $C$ is kept at 100 degrees with steam, while the other half is kept at 0 degrees with ice. Being close to the cold side, we would expect $a$ to be cool. FIGURE 3(b) shows the new temperature distribution obtained by projection through $a$. It is now vividly clear how the distant hot semicircle is 'focused' through $a$ onto a much smaller arc, yielding a low average temperature on $C$ and hence a low temperature at $a$ itself.

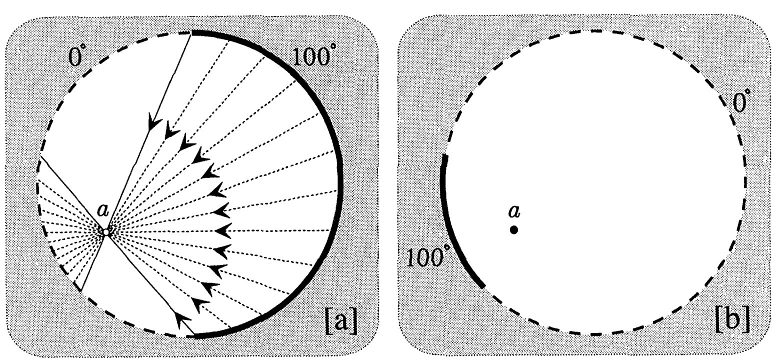


FIGURE 3

While we have not found this approach to Schwarz's result elsewhere, it would be surprising if it had been missed. However, let us end this section by pointing out something that initially obscured the issue for the author, and that may have also hindered other writers. Although we have had no need of it, the *formula* for our geometrically natural mapping is

$$h(z) = R^2\left(\frac{z-a}{\bar{a}z - R^2}\right).$$

For some reason, though, the formula that is conventionally used (e.g., [4], p. 167; [7], p. 197) in the calculational proof of (1) is, instead,

$$k(z) = R^2\left(\frac{z+a}{\bar{a}z + R^2}\right).$$

Of course this too has the required property of mapping 0 to $a$, but it is not self-inverse (as $h$ is) and does not send $a$ to 0.

Since the two mappings are related by $k(z)=h(-z)$, we see that $k(z)$ first projects the points of $C$ through 0, then through $a$. See FIGURE 4(a). Short of this figure itself, there now appears to be no simple relationship among the points $z$, $a$, and $k(z)$.

## 3. Dirichlet's Problem for the Disk

Our example in FIGURE 3 was a trifle hasty. For the moment, Schwarz's result merely says how the interior values of a given harmonic function in the disk may be found from the values of $C$. But in FIGURE 3 we blithely assumed that we could also use it to *construct* such a function in the disk, given arbitrary piecewise-continuous boundary values. In other words, we assumed Schwarz's solution of Dirichlet's problem for the disk (outlined in the introduction). We now justify this.

FIGURE 4(b) shows $a$ approaching a boundary point $z$; also shown are the images ($C_1^*$ and $C_2^*$) under projection through $a$ of the two small arcs ($C_1$ and $C_2$) adjacent to $z$. If the given boundary values are continuous at $z$ then $T$ is essentially constant on $C_1 \cup C_2$, and so the new temperature distribution is likewise almost constant on $C_1^* \cup C_2^*$. As required, the constructed function $T(a)$ therefore *does* approach $T(z)$ as $a$ approaches $z$.

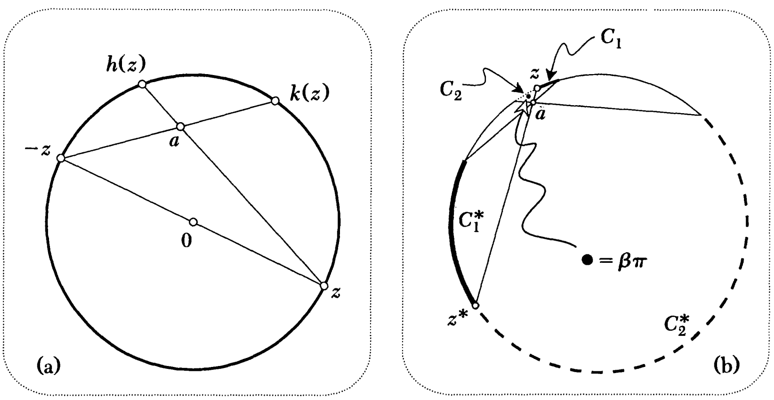


FIGURE 4

Although Dirichlet's problem makes no demands on the behavior of $T(a)$ as $a$ approaches a boundary point at which $T$ is *dis*continuous, it is easy to see (though not to calculate!) what actually happens. Suppose that the boundary temperature jumps from $T_1$ to $T_2$ as we pass from $C_1$ to $C_2$. If $a$ arrives at $z$ while traveling in a direction making an angle $\beta\pi$ with $C_2$, then [exercise] $T(a)$ approaches $[\beta T_1 + (1-\beta)T_2]$. This result is relevant to the representation of discontinuous functions by Fourier series.

It now only remains to show that the constructed function is indeed harmonic. First we shall pause to recover Poisson's formula in its classical form. We begin by noting that (3) may be re-expressed as

$$T(a)=\frac{1}{2\pi}\int_{-\pi}^{\pi} T(\theta)\,d\theta^*. \tag{8}$$

In order to put this into the same form as (1), we now require $d\theta^*$ in terms of $d\theta$. Consider FIGURE 5(a), which shows the movement $R\Delta\theta^*$ of $z^*$ resulting from a movement $R\Delta\theta$ of $z$.

For brevity in this and later arguments, let us employ the following shorthand. If the ratio of $X$ and $Y$ tends to unity as small quantities in a geometric construction tend to zero, we say that $X$ and $Y$ are "ultimately equal", or that $X = Y$ when the small quantities are "infinitesimal." For example, the chord $s$ is ultimately equal to the arc $R\Delta\theta$. It follows from the basic theorems on limits that ultimate equality inherits many of the properties of ordinary equality.

Returning to FIGURE 5(a) the arcs $R\Delta\theta^*$ and $R\Delta\theta$ are ultimately equal to the chords $t$ and $s$, respectively, so $(\Delta\theta^*/\Delta\theta)$ is ultimately equal to $(t/s)$. But $t$ and $s$ are corresponding sides of two similar triangles [shaded], so $(t/s) = (\sigma'/\rho)$. Finally, since $(\sigma'/\rho)$ is ultimately equal to $(\sigma/\rho)$, we obtain

$$\frac{d\theta^*}{d\theta} = \left[\frac{\sigma}{\rho}\right].$$

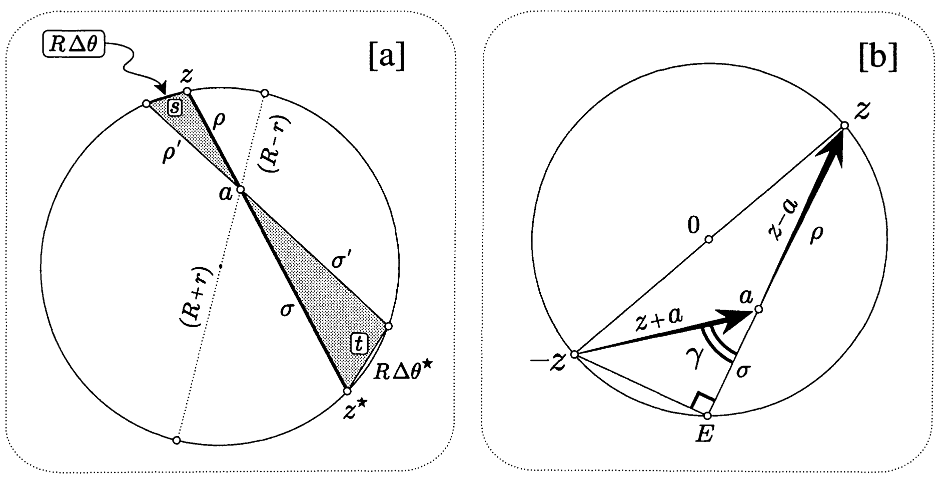


FIGURE 5

Thus (8) becomes

$$T(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{\sigma}{\rho}\right]T(\theta)\,d\theta. \tag{9}$$

Consequently, to derive Poisson's formula we need only show that $[\sigma/\rho]$ is the Poisson kernel $\mathscr{P}_a(z)$. This was precisely how Schwarz [8], working in the opposite direction, originally deducted his result *from* Poisson's formula.

Since $\rho\sigma = \rho'\sigma'$ is constant, we may evaluate it for the dotted diameter through $a$ to obtain $\rho\,\sigma = (R^2 - r^2)$, where $r = |a|$. Thus we do indeed find that

$$\left[\frac{\sigma}{\rho}\right] = \left[\frac{R^2 - r^2}{\rho^2}\right] = \mathscr{P}_a(z).$$

As an interesting consequence of the geometric interpretation of the Poisson kernel, we see that (with $z$ fixed) the level curves of $\mathscr{P}_a$ are the circles that are tangent to $C$ at $z$, with $\mathscr{P}_a = 0$ being $C$ itself.

Returning to the issue of harmonicity, we see that if we permit ourselves differentiation under the integral sign of (9), then it is sufficient to show that $[\sigma/\rho]$ is a harmonic function of $a$. To see that it is, consider FIGURE 5(b). Since the angle at $E$ is a right angle, we have

$$\left[\frac{\sigma}{\rho}\right] = \frac{|z+a|\,\cos\gamma}{|z-a|} = \mathrm{Re}\left(\frac{z+a}{z-a}\right).$$

Because it is the real part of an analytic function of $a$, $[\sigma/\rho]$ is automatically harmonic, and we are done.

This line of reasoning yields a bonus result. Let $S$ be a harmonic conjugate of $T$, so that $f = T + iS$ is an analytic function. This function $f$ is uniquely defined (up to an additive imaginary constant) and so it must be given by

$$f(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{z+a}{z-a}\right)T(\theta)\,d\theta,$$

for this is analytic and has $T(a)$ as its real part. This result is called *Schwarz's formula*, and it enables us to resurrect the complete analytic function $f$ from the ashes of its real part on $C$.

## 4. Hyperbolic Geometry

If we specify arbitrary piecewise-continuous temperatures $T(z)$ along the edge (the real axis) of the upper half-plane, then there is another formula due to Poisson that yields the temperature at any point $a = X + iY$ $(Y > 0)$:

$$T(a) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{Y}{(X-x)^2 + Y^2}\right]T(x)\,dx. \qquad (10)$$

We shall explain this result by reinterpreting (8) in terms of elementary hyperbolic geometry. The transition from (1) to (10) will then be seen as nothing more than a transition between the Poincaré and upper half-plane model of the hyperbolic plane. First, however, let us obtain still another geometric interpretation of Poisson's formula.

For simplicity, let us employ the *unit* circle. Consider FIGURE 6. Let the arc $K$ be heated to unit temperature while the rest of $C$ is kept at zero degrees. By Schwarz's result, the temperature at $a$ is $T(a) = (K^*/2\pi)$, while the temperature at the center of the circle is $T(0) = (K/2\pi)$.

Next, imagine yourself standing at $a$, looking out at a vast number of thermometers placed along the circle. As you turn your head through a full revolution (remembering to turn your feet!) let $\langle T\rangle_a$ denote the average (over all directions) of the temperatures you see. For example, Gauss' mean value theorem may be restated as $T(0) = \langle T\rangle_0$.

In FIGURE 6, $\langle T\rangle_a = (\lambda/2\pi)$, where $\lambda$ is the angle subtended by $K$ at $a$. But we see from the figure that

$$\lambda = \tfrac{1}{2}(K^* + K),$$

so $\langle T\rangle_a = \frac{1}{2}[T(a) + T(0)]$: *The average of the boundary temperatures as they appear to you is equal to the average of the temperature where you are and the temperature at the center*. It is then easy to see that this is still true if we instead have many arcs at different temperatures, and ultimately a general piecewise-continuous temperature distribution. Thus Poisson's formula may be re-expressed as

$$T(a) = 2\langle T\rangle_a - T(0).$$

This result is due to Neumann [9]; we merely rediscovered it, as did Duffin [10] from another point of view. For an interesting generalization, see [11].

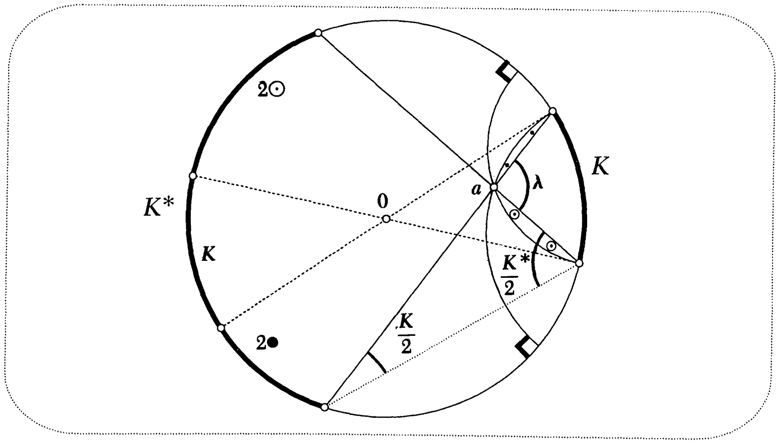


FIGURE 6

FIGURE 7(a) generalizes FIGURE 1(a) and is intended to make this result vivid. Turning your head successively through the same small angle marked • you see the thermometers located at the white dots on the boundary. The average of their temperatures is then a good approximation [exact as $\bullet \rightsquigarrow 0$] to $\langle T \rangle_a$, and hence to the average of the temperature where you stand and the temperature at 0. Note how the white dots became crowded together on the part of the boundary nearest you. As you would expect, this part of the boundary therefore has the greatest influence on the temperature where you stand.

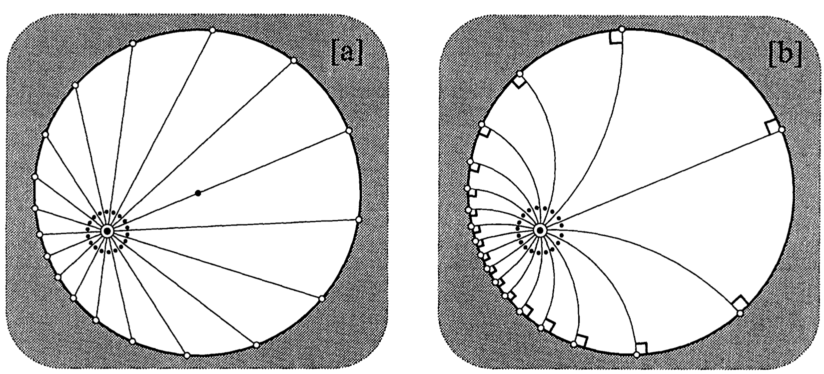


FIGURE 7

To obtain our third and final interpretation of Poisson's formula, let us return to FIGURE 6. Imagine that the disk is the Poincaré model of the hyperbolic plane, and that you are once again standing at the point $a$ looking out to $K$, which is now infinitely far away on the horizon. How big does $K$ appear to you in this distorted geometry? To a godlike observer looking down on this model of the hyperbolic plane, the straight lines along which light travels to you now appear to be arcs of circles orthogonal to $C$, and so *you* see the angular size of $K$ as being

$$\textit{hyperbolic angle} = \lambda + (\bullet + \odot).$$

But we see in the figure that

$$(\bullet + \odot) = \tfrac{1}{2}(K^* - K),$$

and hence we obtain the following remarkable fact:

$$\begin{aligned} \textit{hyperbolic angle} &= \tfrac{1}{2}(K^* + K) + \tfrac{1}{2}(K^* - K) \\ &= K^* \\ &= 2\pi T(a). \end{aligned}$$

The temperature where you are is simply proportional to how big $K$ looks!

Reinterpreting (8), we now see that $d\theta^*$ is simply the hyperbolic angle subtended at $a$ by the element of $C$: *The temperature of each element of C contributes to the temperature at an interior point in proportion to its hyperbolic size as seen from that point*. Much as we did in the Euclidean case, let $\prec T \succ_a$ denote the average of the temperatures you see on the horizon of the hyperbolic plane as you turn your head through a full revolution while standing at $a$. We have found that

$$T(a) = \prec T \succ_a. \tag{11}$$

[Again, we merely rediscovered this: The result (exceeding even the beauty of Schwarz's) is due to Bôcher **[12]**, **[13]**; the only explicit mention of hyperbolic geometry we have found is Carathéodory's **[14]**.] We have chosen to present (11) as a consequence of Schwarz's result, but at the end of the paper we shall see that it can be understood in a much simpler way.

The analogue of FIGURE 7(a) is now FIGURE 7(b). Standing at the same point as before, and again turning your head successively through the angle •, the figure shows the new locations of the thermometers you see on the boundary. The average of their temperatures is then a good approximation [exact as • ⇝ 0] to $\prec T \succ_a$, and hence to the temperature where you stand. Note how the white dots again become crowded together on the part of the boundary nearest you, so that this part of the boundary has the greatest influence on the temperature where you stand.

From the vantage point of (11), the distinction between (2) and (8) evaporates. Every point of the hyperbolic plane is on an equal footing with every other, it is merely that the hyperbolic angle $d\theta^*$ happens to coincide with the more familiar Euclidean angle $d\theta$ when $a = 0$.

Formulated in this way, we may carry the result over to the upper half-plane model for hyperbolic geometry. (The full justification for this transition will be explained at the end of the paper.) See FIGURE 8. The horizon is now the real axis and 'straight lines' are now (for our godlike observer) semicircles meeting the real axis at right angles. The temperature where you stand is now the average (as • ⇝ 0) of the temperatures at the white boundary points in FIGURE 8.

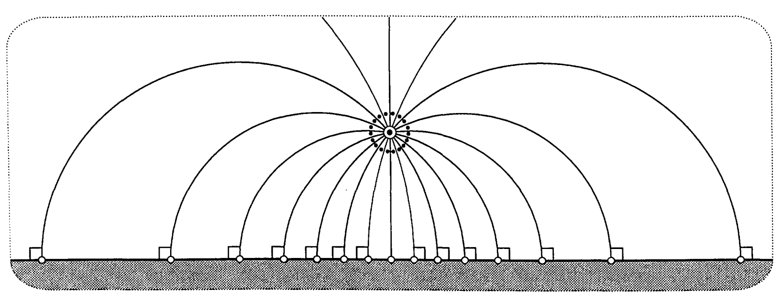

**FIGURE 8**

FIGURE 9 analyzes this in greater detail. It shows both the hyperbolic angle $\Delta\theta^*$ and the Euclidean angle $\Delta\theta$ subtended at $a$ by the element $\Delta x$ of the horizon. Thinking of $\Delta x$ as sufficiently small that $T(x)$ is essentially constant on it, the contribution to the temperature at $a$ is $(1/2\pi)T(x)\Delta\theta^*$. Integrating along the entire horizon we obtain

$$T(a) = \frac{1}{2\pi}\int_{x=-\infty}^{x=\infty} T(x)\, d\theta^*. \tag{12}$$

In order to put this into precisely the same form as (10), we need to find $(d\theta^*/dx)$. We shall do this via an attractive and rather surprising fact: *The non-Euclidean angle $\Delta\theta^*$ is exactly double the Euclidean angle $\Delta\theta$, even if $\Delta x$ is not small.* To see this, concentrate on the semicircle meeting the axis at $p$. The angle between the dotted tangent at $a$ and the vertical is clearly double that between the chord $ap$ and the vertical. The result then follows immediately.

Now consider FIGURE 10. The small shaded triangle is constructed to be right angled, and it is thus ultimately similar to the large shaded triangle as $\Delta\theta$ shrinks to nothing. Thus $(\xi/\Delta x)$ is ultimately equal to $(Y/\Omega)$. Also, since $\xi$ is like a tiny arc of circle of radius $\Omega$, it is ultimately equal to $\Omega\Delta\theta$. Thus if $\Delta\theta$ is infinitesimal,

$$\frac{\Omega\Delta\theta}{\Delta x} = \frac{\xi}{\Delta x} = \frac{Y}{\Omega}.$$

We can now combine this with the previous result to obtain

$$\frac{d\theta^*}{dx} = 2\left[\frac{d\theta}{dx}\right] = 2\left[\frac{Y}{\Omega^2}\right] = 2\left[\frac{Y}{(X-x)^2+Y^2}\right].$$

Putting this into (12), we obtain (10).

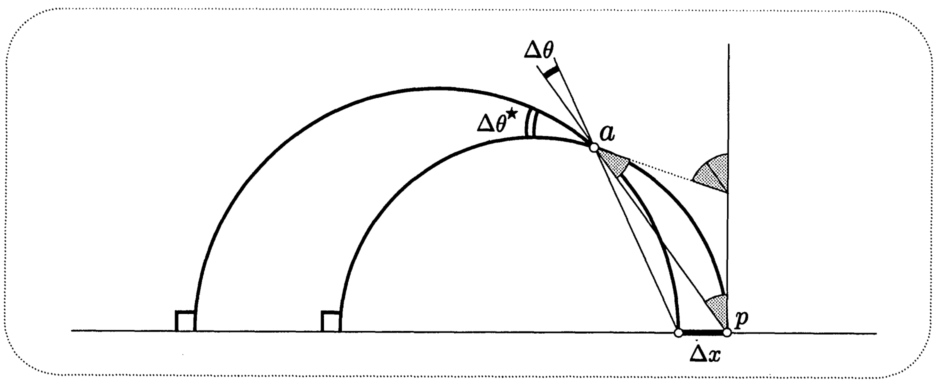


FIGURE 9

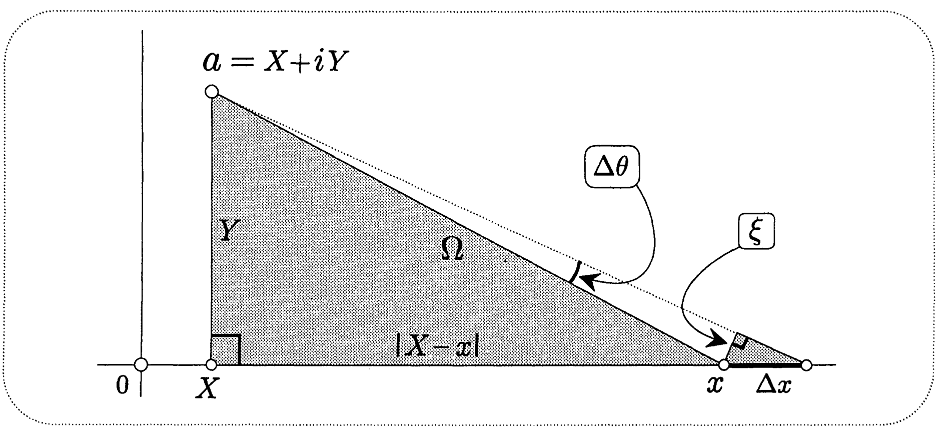


FIGURE 10

While the precise form of the above argument may be new, the basic idea of transferring Bôcher's result from the disk to the half-plane was given by Osgood [15]. For a different but related approach to (10), see [16]. For more on all three of the interpretations thus far obtained, see [11].

## 5. Green's General Formula

Here is the recipe for finding the temperature at the point $a$ inside a simply connected region $R$ in terms of the values $T(z)$ on the boundary $B$.

First, supply heat at the constant rate $2\pi$ to the point $a$ while holding the temperature all round $B$ at the constant value 0. After the heat flow has settled down, the temperature in $R$ will be a well-defined (except at $a$) harmonic function $\mathscr{G}_a(z)$ called the *Green's function* of $R$ with *pole* at $a$. Since $B$ is an isotherm, the heat flow vector $\boldsymbol{H} = -\nabla\mathscr{G}_a$ will be orthogonal to it, and so its magnitude $\mathscr{Q}_a$ (the local heat flux) may be expressed as

$$\mathscr{Q}_a = -\frac{\partial \mathscr{G}_a}{\partial n},$$

where $n$ measures distance in the direction of $\mathbf{N}$, the outward unit normal vector to $B$ (see FIGURE 11).

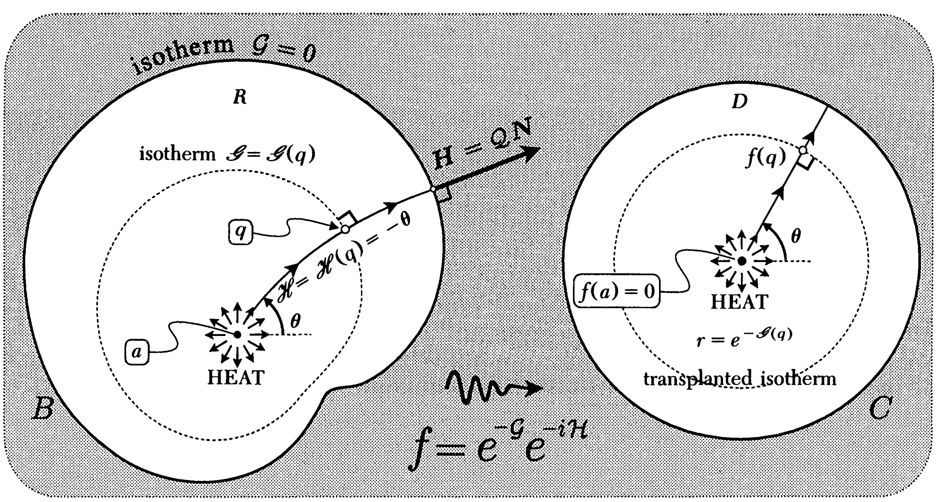


FIGURE 11

Given a harmonic function $T$ on $R$, we may now use $\mathscr{Q}_a$ as a tool with which to find its values inside $R$ in terms of its values $T(z)$ on the boundary $B$. Here is Green's remarkable general formula:

$$T(a) = \frac{1}{2\pi}\oint_B \mathscr{Q}_a(z)T(z)\,ds, \tag{13}$$

where $ds$ is an element of arc length along $B$. Thus $\mathscr{Q}_a$ now plays the same role as the Poisson kernel did in (1).

In order to understand this result we must first understand Riemann's Mapping Theorem: *R may be mapped one-to-one and conformally onto the unit disk D.* We shall see in a moment that the existence of such a mapping is equivalent to the existence of the Green's function. In modern texts the mapping theorem is proved independently of physical considerations, the existence of the Green's function then following as a corollary. It was otherwise for Riemann. He seems to have taken the existence of the Green's function to have been guaranteed by Nature, and he was thereby led to his mathematical mapping theorem. The following is one possibility as to how this may have happened. For a better-motivated approach, see [3].

Let us first consider the behavior of the function $\mathcal{G}_a$ in the immediate vicinity of $a$. Physical intuition leads us to expect, irrespective of the temperatures assigned to $B$, that heat will flow out of the source *symmetrically*: $\mathcal{G}_a$ and $\mathcal{Q}_a$ will be nearly constant on a tiny circle $K$ centered there. If the flow were *perfectly* symmetrical, and if the radius of $K$ were $\rho$, then

$$2\pi = \textit{heat supplied to a} = \textit{flux across } K = 2\pi\rho\mathcal{Q}_a = -2\pi\rho\frac{\partial\mathcal{G}_a}{\partial\rho}.$$

Thus, very close to $a$, the temperature behaves like $-\ln\rho$. The precise statement is that

$$\mathcal{G}_a = -\ln\rho + g_a, \tag{14}$$

where $g_a$ is harmonic throughout $R$.

Let $\mathcal{H}_a$ be a harmonic conjugate of $\mathcal{G}_a$, so that (except at $a$) $\mathcal{F} \equiv \mathcal{G}_a + i\mathcal{H}_a$ is conformal. Since $\ln\rho$ is the real part of $\log(z-a)$ it follows from (14) that $\mathcal{F}(z) = -\log(z-a) + F$, where $F$ is conformal throughout $R$. Since the imaginary part of $F$ is only determined up to a constant, we may choose $\operatorname{Im} F(a) = 0$. Thus, very close to $a$,

$$\mathcal{H}_a = -\arg(z-a).$$

The advantage of this particular choice is that we may then interpret the value of $\mathcal{H}_a$ at a typical point $q$ in the simple manner illustrated in FIGURE 11. Follow the flow of heat back from $q$ to $a$; the angle at which it enters $a$ is then $-\mathcal{H}_a(q)$. We thus have clear physical interpretations for both the real and imaginary parts of the mapping $\mathcal{F}$.

With $\mathcal{F}$ in hand, we are now close to Riemann's theorem. We require a mapping that is conformal throughout $R$, but while the mapping $\mathcal{F}$ is otherwise conformal, it has a *logarithmic singularity* at $a$. In order to undo this singularity, we are therefore *forced* to compose $\mathcal{F}$ with the exponential mapping. The mapping $f$ so obtained is then conformal everywhere in $D$:

$$z \rightsquigarrow w = f(z) = e^{-\mathcal{F}(z)} = e^{-\mathcal{G}_a}e^{-i\mathcal{H}_a}.$$

As illustrated in FIGURE 11, and as was desired, $f$ maps $R$ conformally to $D$: The pole at $a$ maps to 0; the dashed isotherm at temperature $\mathcal{G}(q)$ maps to the dashed circle of radius $e^{-\mathcal{G}_a(q)}$; the streamline $\mathcal{H}_a = \mathcal{H}_a(q) = -\theta$ entering $a$ at angle $\theta$ maps to the ray entering 0 at angle $\theta$.

We make a few further observations before returning to the explanation of (13). Now that we possess the mapping $f$, any harmonic temperature distribution $T(z)$ on $R$ may be *conformally transplanted* to a harmonic function $\tilde{T}(w)$ on $D$ (and vice versa) by assigning equal temperatures to corresponding points of the two regions $\tilde{T}[f(z)] \equiv T(z)$. In particular, the values of $T$ on $B$ are transplanted to $C$.

Next, consider the Green's function of $D$ with pole at 0. See FIGURE 11. On grounds of symmetry, the isotherms must be circles centered at 0, and the streamlines must be rays emanating from there. More precisely, it should be clear that if $w$ denotes a point of $D$ then the Green's function is $-\ln|w|$. But this means [make sure you can see this] that each point of $D$ is at the same temperature as its preimage in $R$; in other words, $f$ conformally transplants the Green's function $\mathcal{G}_a(z)$ of $R$ with pole at $a$ to the Green's function of $D$ with pole at $f(a) = 0$.

This is no accident. More generally, let $J(z)$ be a one-to-one conformal mapping of $R$ to some other simply connected region $S$ with boundary $Y$. Then $J$ conformally transplants the Green's function $\mathscr{G}_a(z)$ of $R$ with pole at $a$ to the Green's function of $S$ with pole at $J(a)$. In particular, the streamlines of the flow in $R$ map to the streamlines of the flow in $S$. In this sense, *the concept of the Green's function is conformally invariant.*

The explanation is not difficult. Since $\mathscr{G}_a$ is harmonic except at $a$, its transplant is harmonic except at $J(a)$. Also since $\mathscr{G}_a$ vanishes on $B$, its transplant vanishes on $Y$. To complete the proof, we must show that the source at $a$ transplants to a source of equal strength at $J(a)$. To see this, recall that the local effect of an analytic function $J$ is an expansion by $|J'|$ and a rotation of $arg(J')$. Geometrically, the source of strength $2\pi$ at $a$ is characterized by the fact that the radii of the infinitesimal circular isotherms round $a$ are proportional to $e^{-temperature}$. The mapping $J$ merely expands these by $|J'(a)|$ to produce infinitesimal circular isotherms round $J(a)$ having (by definition) the same temperatures as the originals. Thus the radii of the transplanted isotherms round $J(a)$ are again proportional to $e^{-temperature}$, so we have a source of strength $2\pi$ at $J(a)$. Done.

We are now in a position to explain the general formula in a beautifully simple way. To begin with, imagine that $T(z)$ is a given harmonic function in $R$ whose value $T(a)$ at an interior $a$ we wish to determine from the boundary values. See FIGURE 12. Just as in FIGURE 7(a), imagine standing inside $R$ at $a$ and turning your head successively through the small angle •. But now *suppose that light travels along the illustrated streamlines of the heat flow* $\boldsymbol{H} = -\nabla\mathscr{G}_a$ *associated with the Green's function.* You would then see the thermometers at the illustrated points on the boundary.

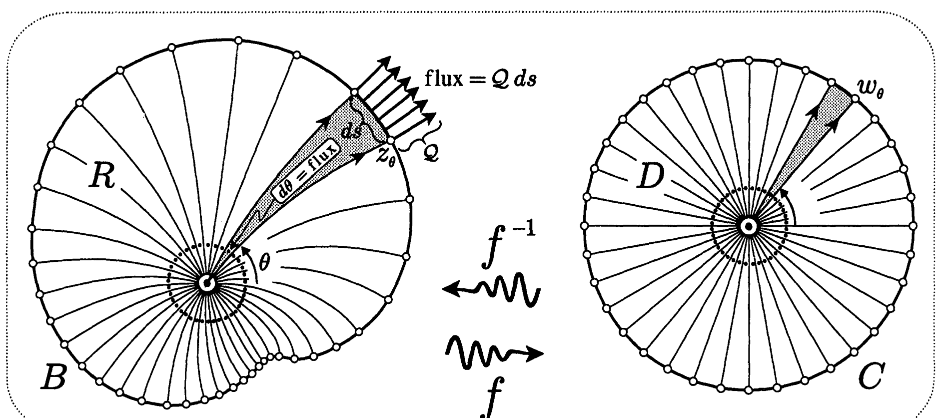


FIGURE 12

The key observation is that (even without passing to the limit of vanishing •) *the average of the observed temperature is conformally invariant.* As before, let $J(z)$ be a one-to-one conformal mapping of $R$ to some other simply connected region $S$ with boundary $Y$. Just as we did with $f$, let us choose $J$ so that the directions of curves through $a$ are preserved (i.e., $\arg[J'(a)] = 0$). Let $z_\theta$ denote the point on $B$ that you see when you look in the direction $\theta$, and let $w_\theta \equiv J(z_\theta)$ be its image on $Y$.

By the conformal invariance of the Green's function, the image of the streamline leaving $a$ at angle $\theta$ is the streamline leaving $J(a)$ at the same angle. Thus $w_\theta$ is not only the image of $z_\theta$, it is also the boundary point which an observer at $J(a)$ sees when looking in the direction $\theta$. But, by definition, the temperature at each point $z_\theta$

on $B$ is transplanted to $w_\theta$ on $Y$, so the observer $J(a)$ sees exactly the same temperatures on $Y$ as the original observer at $a$ saw on $B$.

Passing to the limit of vanishing •, the conformal invariance of this average may be expressed as

$$\frac{1}{2\pi}\oint_B T(z_\theta)\,d\theta = \frac{1}{2\pi}\oint_Y \tilde{T}(w_\theta)\,d\theta.$$

FIGURE 12 illustrates the particular case where $J = f$ is the previously constructed function that maps $R$ to $D$ and $a$ to 0. The virtue of this special case is that the conformally invariant average may now be *evaluated*. By Gauss' mean value theorem, the average of $\tilde{T}(w_\theta) \equiv T(z_\theta)$ on $C$ is the temperature $\tilde{T}(0) \equiv T(a)$ at the center:

$$\frac{1}{2\pi}\oint_B T(z_\theta)\,d\theta = \frac{1}{2\pi}\oint_C \tilde{T}(w_\theta)\,d\theta = T(a).$$

Thus, returning to the left-hand side of FIGURE 12, we have found that in the limit of vanishing • *the average of the temperatures you see on the boundary of R is the temperature where you stand!*

It now remains to show that this geometric construction is equivalent to the conventional formula (13). On the left of FIGURE 12, consider the shaded region—let us call it a "tube"—between two of the streamlines leaving $a$ with infinitesimal angular separation $d\theta$, and let the tube intercept $B$ at $z_\theta$ in an element of length $ds$. Since $2\pi$ of heat flux emerges symmetrically from $a$, the flux emitted into the tube is equal to $d\theta$. Furthermore, since $\mathcal{G}_a$ is harmonic, no heat is created or destroyed in the tube, so all the heat that enters the tube at the source must emerge at the other end. Thus $d\theta = \mathcal{Q}_a(z_\theta)\,ds$, and

$$T(a) = \frac{1}{2\pi}\oint_B T(z_\theta)\,d\theta = \frac{1}{2\pi}\oint_B \mathcal{Q}_a(z)T(z)\,ds,$$

as was to be shown. Surprisingly, we have not found this interpretation and explanation elsewhere in the literature. We hope you will agree that it contrasts strikingly with the conventional approach (c.f. [7], p. 209).

This line of reasoning also explains the stronger result that (13) solves Dirichlet's problem for $R$. Using $f$ to conformally transplant the given boundary values from $B$ to $C$, we know that Poisson's formula allows us to construct the solution to Dirichlet's problem in $D$. Transferring this solution back from $D$ to $R$ with $f^{-1}$, we have found the harmonic function $T$ in $R$, and its value at $a$ must then be given by (13). You can now understand why we lavished so much attention on the special case of the disk.

We can also use this conformally invariant average to understand much of what has gone before in a simple, unified way. For example, the average $\prec T \succ_a$ depicted in FIGURE 7(b) is merely the special case in which $R$ is the unit disk. To see this, recall (7). The conformal mapping $h(z)$ of $D$ to itself interchanges 0 and $a$ and maps circles to circles. It follows [why?] that it interchanges FIGURE 7(b) and FIGURE 1(a). Similarly by employing an inversion in a circle centered on the real axis it is possible (see [5]) to construct an analogous (conformal and circle-preserving) mapping from the upper half-plane to the unit disk. It follows that the result in FIGURE 8 is also a special case.

Although we hope you will agree that this is all delightfully intuitive, one could still wish for an explanation of (13) that dealt directly with $R$ rather than requiring the assistance of the disk. It does not appear to be widely known, but such an explanation is indeed possible. Working with electricity rather than heat, James Clerk Maxwell [17] was able to give a direct explanation of (13) by arguing in terms of electrostatic energy. But because the required reasoning is *purely* physical, we shall not enter into this matter here.

**Acknowledgments.** I should like to thank Prof. Roger Penrose for suggesting future lines of investigation in this area, and Dr. George Burnett-Stuart for explaining to me Maxwell's splendid work on this subject. I also thank the Faculty Development Committee of the University of San Francisco for making possible the trip to Oxford during which the conversations with the above gentlemen took place.

## REFERENCES

1. W. Zimmermann, S. Cunningham (Editors), *Visualization in Teaching and Learning Mathematics*, MAA Notes 19, Washington, DC, 1991.
2. T. Needham, Newton and the transmutation of force, *Amer. Math, Monthly*, **100** (1993), 119–137.
3. T. Needham, *Visual Complex Analysis*, Oxford University Press, Oxford, (to appear).
4. L. Ahlfors, *Complex Analysis*, 3rd edition, McGraw-Hill Book Co., New York, 1979.
5. J. Stillwell, *Geometry of Surfaces*, Springer-Verlag, New York Inc., New York 1992.
6. W. Thurston, *Three-Dimensional Geometry and Topology*, draft available from the Geometry Center, University of Minnesota.
7. R. Boas, *Invitation to Complex Analysis*, Random House, Inc., New York, 1987.
8. H. A. Schwarz, *Gesammelte Mathematische Abhandlungen*, Vol. 2, Chelsea Publishing Co., Inc., New York, 1972, p. 360.
9. C. Neumann, *Vorlesungen über Riemann's Theorie der Abel'schen Integrale*, 2nd edition, B G. Teubner, 1884, p. 410.
10. R. J. Duffin, A note on Poisson's integral, *Quarterly of Applied Math.*, **15** (1957), 109–111.
11. F. W. Perkins, An intrinsic treatment of Poisson's integral, *Amer. Jour. of Math.*, **50** (1928), 389–414.
12. M. Bôcher, Note on Poisson's Integral, *Bull. Amer. Math. Soc.*, **4** (1898), 424–426.
13. M. Bôcher, On harmonic functions in two dimensions, *Proc. Amer. Acad. Arts & Sci.*, **41** (1906), 577–583.
14. C. Carathéodory, *Theory of Functions of a Complex Variable*, Vol. 1, Chelsea Publishing Co., 1954, p. 156.
15. W. F. Osgood, *Lehrbuch der Funktionentheorie*, 5th edition, Vol. 1, Chelsea Publishing Co., 1928, pp. 666–670.
16. R. Lange, R. A. Walsh, A heuristic for the Poisson integral for the half-plane and some caveats, *Amer. Math. Monthly*, **92** (1985), 356–358.
17. J. C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd edition, Vol. 1, Dover Publications, Mineola, NY, 1954, pp. 103–107.

## 50 YEARS AGO...

Grace Chisholm Young died just before she was to receive an honorary degree from the Fellows of Girton College, Cambridge. Grace Chisholm was born near London in 1868, was educated at home, and entered Girton, the first English institution to allow women to receive a university education. She attained a superior score on the Cambridge Tripos and continued her education at Göttingen, earning her Ph.D. in 1895—the first woman to receive a German doctorate in mathematics through the regular procedure!

from Victor J. Katz, *A History of Mathematics*, Harper Collins, 1993.