ARTICLES

Cube Slices, Pictorial Triangles, and Probability

DON CHAKERIAN
University of California
Davis, CA 95616

DAVE LOGOTHETTI
Santa Clara University
Santa Clara, CA 95053

1. Introduction

If we slice a cube by planes perpendicular to a body diagonal and passing through vertices, we see in four successive positions cross sections consisting of a point, a "rightside-up" triangle, an "upside-down" triangle, and another point (FIGURE 1). The numerical pattern of the vertices in these cross sections is 1, 3, 3, 1, famous for being a row of Pascal's triangle.

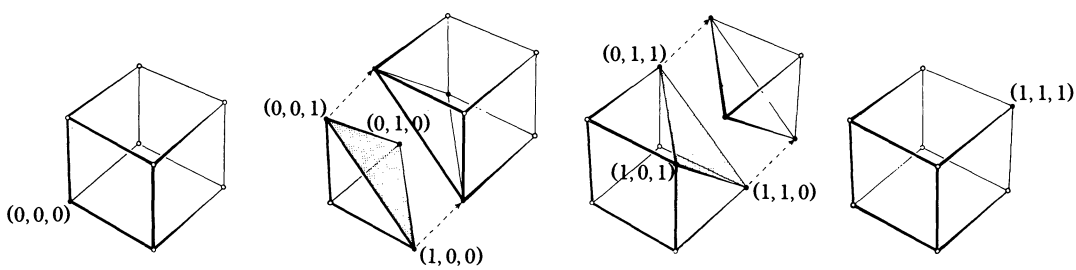


FIGURE 1

Backing down to a 2-dimensional cube, alias a "square," we see three successive cross sections consisting of a point, a line segment, and another point, with the vertices generating the numerical pattern of 1, 2, 1, another row of Pascal's triangle. (See FIGURE 2.) And backing down to 1-dimensional and 0-dimensional cubes ("line segment" and "point"), we analogously find the top two rows of Pascal's triangle.

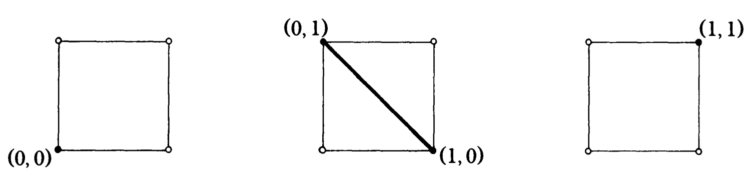


FIGURE 2

We gather these results together in FIGURE 3, showing for $k = 0, 1, 2, 3$, successive slices of the k-dimensional cube and the numbers of vertices on the cross sections.

In this article we pursue this pattern into higher dimensions, showing that successive cross sections of an n-dimensional cube along a body diagonal and passing through lattice points on hyperplanes perpendicular to this diagonal give rise to a family of figures that can be pictorially generated in a manner analogous to the way the binomial coefficients are generated in Pascal's triangle. In Section 2 we describe how the pictorial analogue of Pascal's triangle in FIGURE 3 continues quite congenially into dimensions $4, 5, 6, \ldots$, with numbers of vertices on successive cross sections reproducing the pattern in the nth row of Pascal's triangle (where by convention the top row is the "0-th" row). For example, the successive cross sections perpendicular to a body diagonal of a 4-dimensional cube and containing vertices of the cube are a point, a "rightside-up" tetrahedron, an octahedron, an "upside-down" tetrahedron, and another point, with respective numbers of vertices 1, 4, 6, 4, 1 (as illustrated on the bottom row of FIGURE 5).

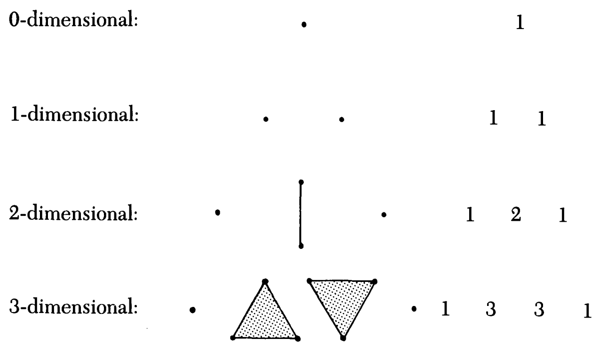


FIGURE 3

Not only do the figures in this pictorial version of Pascal's triangle have numbers of vertices corresponding to the entries in the usual Pascal's triangle, but *each of these figures also can be got by combining the two figures directly above it in a pictorial analogy to Pascal's identity,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

In fact, each figure is generated by forming the convex hull of the union of appropriately positioned copies of the two figures in the immediately preceding row. By the "convex hull" of a set S we mean the smallest convex set containing S, i.e., the intersection of all convex sets containing S. The convex hull of a finite collection of points is a closed and bounded convex polyhedron, which we shall call a "polytope." (See Lay **[12]** or Grünbaum **[7]** for a detailed discussion of these matters.) These cross sections of the n-dimensional cube constitute a family of polytopes, all but two being $(n-1)$-dimensional (with the first and last 0-dimensional). We denote these cross-sectional polytopes $P(n,k)$, $k = 0, 1, 2, \ldots$, so each $P(n,k)$ has

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

vertices. Thus $P(4,2)$ has six vertices and turns out, satisfyingly enough, to be a regular octahedron. We discuss the structure of $P(n,k)$ in Section 3. This recursive procedure for drawing the pictures of the $P(n,k)$ (and other clusters of lattice points in later sections) is reminiscent of Pólya's "picture writing" (Pólya **[15]**).

In Section 4 we deal with an n-dimensional cube of edgelength 2, subdivided into unit cubes. Successive slices by planes perpendicular to a body diagonal and passing through subdivision vertices give rise to a numerical triangle of special trinomial coefficients, a companion of Pascal's triangle (Pólya **[16**, p. 87**]**) that has in turn a pictorial triangle of its own, illustrated in FIGURE 10.

The vertices of our cubes lie on "lattice points" in n-dimensional space: just those points with integer coefficients. In Section 5 we count the lattice points of an m-by-m-by-$\cdots$-by-m cube in n-dimensional space that lie on one of our slicing planes and obtain further numerical and pictorial triangles of combinatorial interest. It turns out that these numbers of lattice points appear as coefficients of the expansion of $(1+t+t^2+\cdots+t^m)^n$ into powers of t. In particular, for $m=1$ we get the usual binomial coefficients in the expansion of $(1+t)^n$, while for $m=2$ we get the trinomial coefficients in the expansion of $(1+t+t^2)^n$, which we explore in Section 4. See FIGURES 12 and 13 for companion numerical and pictorial triangles corresponding to $m=3$.

By counting lattice points on a slice, multiplying by an appropriate factor, and taking a limit, we derive in Section 6 a formula for the $(n-1)$-dimensional volumes (to which we informally refer as "areas") of cube slices perpendicular to a body diagonal. This formula has a longish history and familiar names associated with it, including those of Laplace and Pólya. We apply the formula to the especially interesting central slices connected with a variety of geometric problems, several of which we discuss in Section 11.

In Section 7 we integrate slice areas to get volumes, particularly volumes of slabs of a cube, and these generate further numerical triangles, with Eulerian numbers and their close relatives, Slepian numbers, suddenly appearing on the scene.

We use a volume formula in Section 8 to solve a Putnam problem, and we show in Sections 9 and 10 how these area and volume calculations provide solutions to a pair of appealing problems in geometric probability. For those who wish to investigate geometric interpretations of combinatorial arrays, we recommend the article of Putz [17], wherein geometry and combinatorics are connected differently.

2. Cube slices and the pictorial triangle

In order to transcend the inadequacy of our 3-dimensional vision and obtain a clearer view of cross sections of cubes with dimension greater than three, we introduce a coordinate system and don the spectacles of analytic geometry.

Consider the cube in standard position in n-dimensional Euclidean space, with vertices at the 2^n points with coordinates $(x_1, x_2, \ldots, x_n)$, where each x_i is 0 or 1, to which we will refer as the "unit cube." We want to slice this cube with hyperplanes perpendicular to the diagonal joining $(0,0,\ldots,0)$ to $(1,1,\ldots,1)$. Each such hyperplane has an equation $x_1+x_2+\cdots+x_n=t$, with t varying from 0 to n as we move from the hyperplane through $(0,0,\ldots,0)$ to the parallel hyperplane through $(1,1,\ldots,1)$. Thus, each hyperplane is a translate of the $(n-1)$-dimensional subspace perpendicular to the body diagonal. We shall informally refer to these hyperplanes as "planes" from now on.

These coordinates shed an explanatory light on why Pascal's triangle popped out at us. The plane $x_1+x_2+\cdots+x_n=0$ contains only the vertex $(0,0,\ldots,0)$, while the plane $x_1+x_2+\cdots+x_n=1$, for example, contains the n vertices (lattice points) of the cube having exactly one coordinate 1 and the others 0. In general, the plane $x_1+x_2+\cdots+x_n=k$, $k=0,1,\ldots,n$, contains those vertices $(x_1,x_2,\ldots,x_n)$ having exactly k coordinates equal to 1 and the rest 0. Since each such vertex is determined by choosing k positions for the 1's from the n coordinate slots available, the number of these vertices is $\binom{n}{k}$, read "n choose k." So if we denote the kth cross section by $\mathrm{P}(n,k)$, $1 \le k \le n-1$, then $\mathrm{P}(n,k)$ is an $(n-1)$-dimensional polytope with $\binom{n}{k}$ vertices, while $\mathrm{P}(n,0)$ and $\mathrm{P}(n,n)$ are the points $(0,0,\ldots,0)$ and $(1,1,\ldots,1)$, respectively.

FIGURE 1 illustrates the case for $n = 3$. Reverting to the usual notation in Euclidean 3-space, we see the four successive slices of the unit 3-cube by the planes with equations $x + y + z = k$, for $k = 0, 1, 2, 3$. So, for instance, P(3, 2) is the triangle with the $\binom{3}{2}$ vertices (0, 1, 1), (1, 0, 1), and (1, 1, 0) determined by a slice of the plane $x + y + z = 2$.

The essence of mathematics is suspicion (with extreme paranoia yielding the best results), and readers should perhaps have a nagging doubt about our assertion that $P(n, k)$ has exactly $\binom{n}{k}$ vertices. We know that the $\binom{n}{k}$ vertices of the cube lying in the plane $x_1 + x_2 + \cdots + x_n = k$ are indeed vertices of the cross section, but it is not completely obvious that this polytope might not have some other vertices. To see, in fact, that $P(n, k)$ has no vertices that are not already vertices of the n-cube, it suffices to note that any vertex of a cross section is precisely where the plane meets a 1-dimensional edge of the cube, so all we need to show is that our plane meets no interior point of an edge. This is easily achieved by mathematical induction; alternatively, we can determine explicitly where a plane meets an edge, which we do in the next section.

The fact that $P(n, k)$ has exactly $\binom{n}{k}$ vertices tells little about its structure in general. We now demonstrate how $P(n + 1, k)$ can be constructed from $P(n, k)$ and $P(n, k - 1)$ in a natural manner completely analogous to Pascal's identity for generating binomial coefficients,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

In order to do this, let us take C to be the unit $(n + 1)$-cube (one dimension higher than heretofore) and H to be the plane with equation $x_1 + x_2 + \cdots + x_{n+1} = k$, $1 \le k \le n$. Then $H \cap C = P(n + 1, k)$, whose vertices fall into two sets: the first lying in F_0, the "bottom" face of C where the final coordinate equals 0, and the second lying in F_1, the "top" face of C where the final coordinate equals 1. Observe that F_0 is a standard unit n-cube, and $H \cap F_0$ consists of those points $(x_1, x_2, \ldots, x_n, 0)$ such that $x_1 + x_2 + \cdots + x_n = k$; i.e., $H \cap F_0$ is a copy of $P(n, k)$. Similarly, $H \cap F_1$ is a copy of $P(n, k - 1)$. Those vertices of $P(n + 1, k)$ belonging to F_0 are precisely the $\binom{n}{k}$ vertices of $H \cap F_0$, and the vertices of $P(n + 1, k)$ belonging to F_1 are the $\binom{n}{k-1}$ vertices of $H \cap F_1$. FIGURE 4 is a schematic picture of what we have here, with $n = 2$ and $k = 2$.

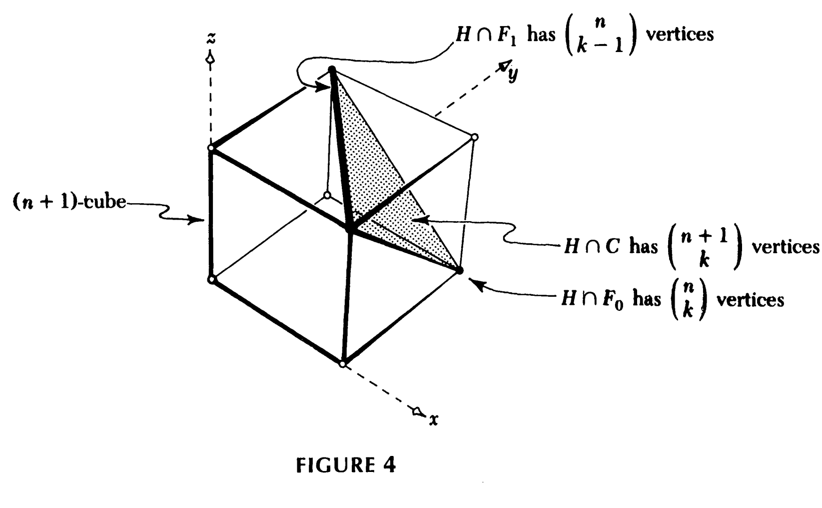


FIGURE 4

Since $P(n+1, k)$ is the convex hull of the set of its vertices, it follows that it is the convex hull of the union of $H \cap F_0$ and $H \cap F_1$. Thus, $P(n+1, k)$ is the convex hull of two appropriately positioned copies of $P(n, k-1)$ and $P(n, k)$.

We repeat ourselves, since this is in a nutshell the main idea of this paper: While in Pascal's triangle we have

$$\begin{matrix} \binom{n}{k-1} & & \binom{n}{k} \\ \searrow & & \swarrow \\ & \binom{n+1}{k} & \end{matrix}$$

meaning we obtain $\binom{n+1}{k}$ by taking the sum of $\binom{n}{k-1}$ and $\binom{n}{k}$, in our pictorial triangle we have

$$\begin{matrix} P(n, k-1) & & P(n, k) \\ \searrow & & \swarrow \\ & P(n+1, k) & \end{matrix}$$

meaning we obtain $P(n+1, k)$ by taking the convex hull of appropriately positioned copies of $P(n, k-1)$ and $P(n, k)$. FIGURE 5 portrays the first five rows of our pictorial triangle, adding one more row to that pictured in FIGURE 3. The last row in this figure gives the cross sections of the unit 4-cube (also called a "tesseract") by the planes $x_1 + x_2 + x_3 + x_4 = k$, for $k = 0, 1, 2, 3, 4$. In the center of that row we have $P(4, 2)$, which is a regular octahedron, the convex hull of the properly positioned (in parallel planes) equilateral triangles $P(3, 1)$ and $P(3, 2)$ from the preceding row. Thus, *the central section of a 4-dimensional cube is a 3-dimensional regular octahedron*.

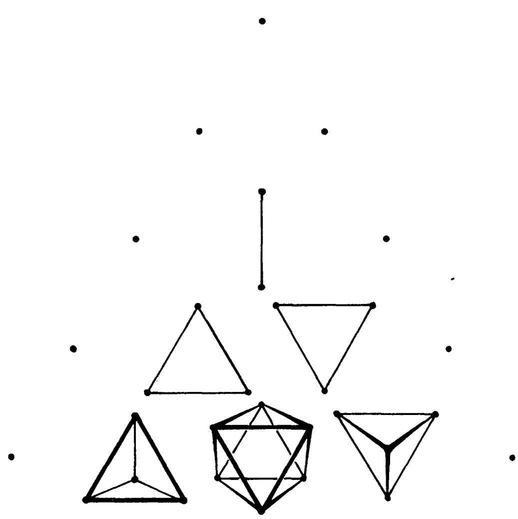


FIGURE 5

The plane $x_1 + x_2 + x_3 + x_4 = 2$ passes through the center $(1/2, 1/2, 1/2, 1/2)$ of the unit 4-cube. Therefore, the regular octahedron $P(4, 2)$ is a "central slice" of the 4-cube, perpendicular to the main diagonal. Such central slices of n-cubes will snag our attention later.

Drawing the next row of the pictorial triangle in FIGURE 5, corresponding to $n = 5$, would require pictures of six cross sections of the 5-cube, four of which are 4-dimensional and too scary to sketch, although we shall describe them in the following section. We can see at least that for any n the polytopes $P(n,1)$ and $P(n,n-1)$ are regular $(n-1)$-simplices. A regular $(n-1)$-dimensional simplex is the convex hull of n points whose mutual distances apart are equal, this common distance being the "edgelength" of the simplex. Inquisitive readers may check via coordinates of vertices that $P(n,1)$ and $P(n,n-1)$ have edgelength $\sqrt{2}$, for $n = 2,3,4,\ldots$.

3. Another look at cube slices

We want to get on with the combinatorial and probabilistic aspects of cube slicing; however, it will be useful to pause for an examination of the structure of the polytopes $P(n,k)$ from a new viewpoint. In later sections we will deal with *general* cross sections of an n-cube perpendicular to its main diagonal, not necessarily those containing vertices of the cube. For instance, a central slice of a 3-cube perpendicular to its main diagonal is a regular hexagon containing none of the vertices of the cube. FIGURE 6 shows such a slice of the unit 3-cube by the plane $x+y+z=3/2$.

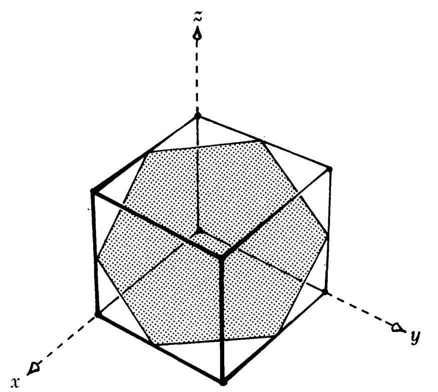


FIGURE 6

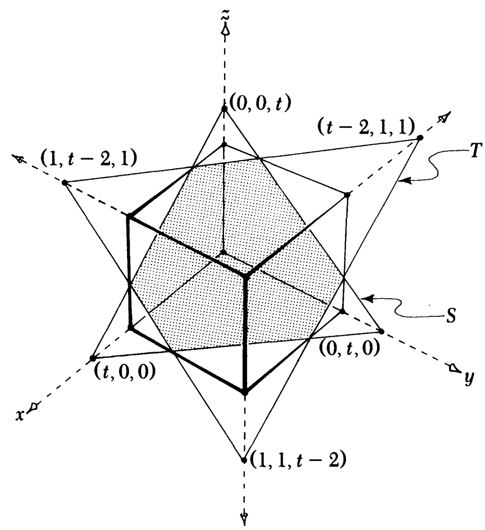


FIGURE 7

It turns out that any slice of an n-cube perpendicular to a main diagonal may be viewed as the intersection of two oppositely oriented regular $(n-1)$-dimensional simplices having a common centroid but possibly different edgelengths. The regular hexagon in FIGURE 6 is the intersection of two oppositely oriented equilateral triangles sharing the same centroid. These triangles happen to have the same edgelength since the slice is through the center of the cube.

Equivalently, we can describe each polytope as a truncated regular $(n-1)$-simplex, with the vertices amputated by planes equidistant from and parallel to the opposite faces. This structure may strain our imaginations when amputated portions overlap.

Here we sketch the 3-dimensional case and indicate why the result holds in n-dimensions. FIGURE 7 shows a slice of the unit 3-cube C by a plane H with equation $x+y+z=t$, $0<t<3$. Triangle S with vertices $(t,0,0)$, $(0,t,0)$, $(0,0,t)$ lies in H, and the slice $H \cap C$ is just S after its vertices have been amputated by faces of C. So $S \cap C = H \cap C$. Now note that H contains the points $(t-2,1,1)$, $(1,t-2,1)$, $(1,1,t-2)$, and hence triangle T with these points as vertices satisfies $T \cap C = H \cap C$; therefore, slice $H \cap C$ is the same as $S \cap T$. Also observe that triangles S and T have their common centroid (the average of their vertices) at $(t/3, t/3, t/3)$.

All of this can be generalized to n-dimensions, with C the unit n-cube, H the plane with equation $x_1 + x_2 + \cdots + x_n = t$, $0 \le t \le n$, S the regular $(n-1)$-dimensional simplex with vertices $(t, 0, \ldots, 0), (0, t, \ldots, 0), \ldots, (0, 0, \ldots, t)$, and T the regular simplex with vertices $(t-n+1, 1, \ldots, 1), (1, t-n+1, \ldots, 1), \ldots, (1, 1, \ldots, t-n+1)$. These simplices are oppositely oriented and have their common centroid at $(t/n, t/n, \ldots, t/n)$, with S's edgelength $t\sqrt{2}$ and T's edgelength $(n-t)\sqrt{2}$.

A central slice corresponds to $t = n/2$, in which case both simplices have the same edgelength $n/\sqrt{2}$. For example, the regular octahedron P(4,2) at the bottom of FIGURE 5 is a central slice of the unit 4-cube. The edgelength of this octahedron is $\sqrt{2}$, and the octahedron is the intersection of two oppositely oriented regular tetrahedra with edgelengths $2\sqrt{2}$ and a common centroid. (See also FIGURE 15.)

The preceding shows that $P(n,k)$ may be viewed as the intersection of a regular $(n-1)$-dimensional simplex of edgelength $k\sqrt{2}$ with an oppositely oriented simplex having the same centroid and edgelength $(n-k)\sqrt{2}$. In case $k = 1$ or $n-1$, the intersection will be simply the smaller of the two simplices.

Coxeter [4] discusses the polytopes $P(n,k)$, using the following notation for truncations:

$$P(n,k) = \left\{ \begin{matrix} 3^{k-1} \\ 3^{n-k-1} \end{matrix} \right\},$$

and he points out [4, p. 239] how their vertices are distributed among the vertices of a cube.

4. The 2-by cube and trinomial coefficients

We now double the edgelength of our cube and cut it into smaller unit cubes. The doubled n-dimensional cube we take to have vertices $(x_1, x_2, \ldots, x_n)$, where each x_i is either 0 or 2. This is a "2 by 2 by... by 2" cube, to which we will refer simply as a "2-by cube." This 2-by cube we cut into 2^n unit cubes with n planes parallel to the faces. The vertices of the little cubes are precisely the lattice points contained in the 2-by cube. There are 3^n lattice points belonging to the n-dimensional 2-by cube, namely those $(x_1, x_2, \ldots, x_n)$ with each x_i equal to 0, 1, or 2. In FIGURE 8 we see the 3-dimensional 2-by cube partitioned into $2^3 = 8$ unit cubes, determining $3^3 = 27$ lattice points.

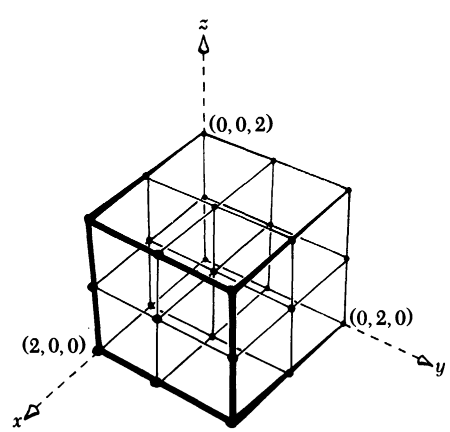


FIGURE 8

We slice the 2-by cube with a plane perpendicular to the main diagonal again, this time counting the number of *lattice points* (rather than *vertices*) in the cube belonging to the plane. Since such a plane laden with lattice points has the equation

$x_1 + x_2 + \cdots + x_n = k$, for some $k = 0, 1, \ldots, 2n$, what we really are doing is counting the number of integral solutions of this equation, with $0 \leq x_i \leq 2$.

In FIGURE 9 we display the successive slices of the 2-by cube of FIGURE 8 by the planes $x + y + z = k$, $k = 0, 1, \ldots, 6$, showing in each case the lattice points on every slice. The corresponding numbers of lattice points are 1, 3, 6, 7, 6, 3, 1, with a sum of 27.

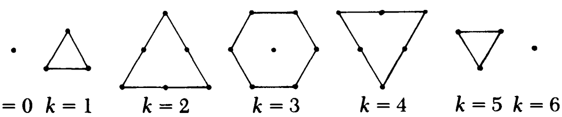


FIGURE 9

In FIGURE 10 we show, analogously to FIGURE 3, the 2-by cubes of dimensions 0, 1, 2, 3, the pictorial triangle of cross sections with lattice points, and the corresponding numerical triangle of numbers of lattice points on each slice. (We will shortly show the significance of the dotted "tees"; in the meantime, we trust these dotted tees will not cause crossed eyes.)

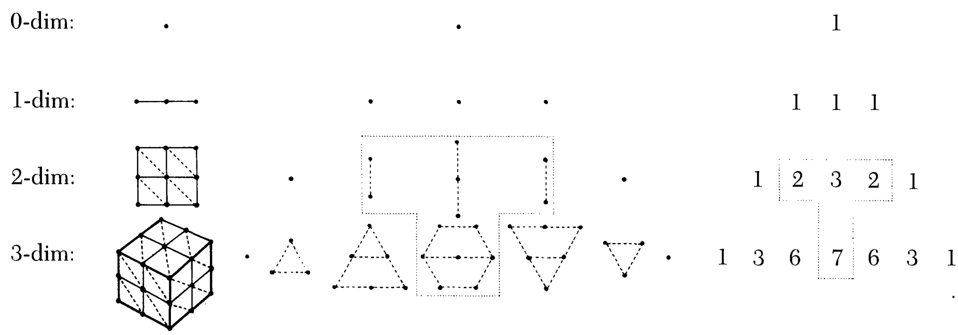


FIGURE 10

In FIGURE 10 we have the beginning of a triangle of trinomial coefficients, those that appear in the expansion of $(1 + t + t^2)^n$ into powers of t. For example, $(1 + t + t^2)^3 = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + t^6$, the coefficients of which form the bottom row of the numerical triangle in FIGURE 10, corresponding to dimension $n = 3$.

The generating rule for the numerical triangle of trinomial coefficients resembles the rule for Pascal's triangle, in that each number is the sum of the nearest *three* numbers in the preceding row. This follows directly from the identity, $(1 + t + t^2)(1 + t + t^2)^n = (1 + t + t^2)^{n+1}$. Readers may wish to check that the next row in the trinomial triangle is 1 4 10 16 19 16 10 4 1. Note that in the generating rule a 0 is used for missing numbers at the ends of the preceding row.

As for pictures, the entries of the pictorial triangle in FIGURE 10 are generated analogously, with "convex hull" replacing "sum," just as in the case of the pictorial analogue of Pascal's triangle. For instance, a triangle of lattice points in the last row is obtained by forming the convex hull of appropriately positioned copies of the three figures above it; similarly, the dotted tee shows that the hexagon of 7 lattice points in the middle of the last row is the convex hull of the three clusters above it, corresponding to $7 = 2 + 3 + 2$ in the numerical triangle.

The figure that should appear directly below the hexagon in the pictorial triangle will be one with 19 lattice points, corresponding to the trinomial coefficient in the same place in the numerical triangle. This figure is obtained by slicing the 4-dimensional 2-by cube with the plane $x_1 + x_2 + x_3 + x_4 = 4$. This slice is, again, a regular octahedron, and the resulting cluster of lattice points is generated via the rule for the pictorial triangle as illustrated in FIGURE 11.

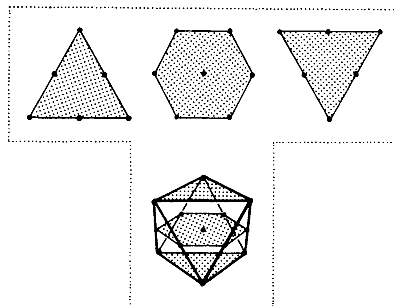


FIGURE 11

5. Slicing general m-by cubes

Let m be a non-negative integer; we shall refer to the m by m by...by m cube with vertices $(x_1, x_2, \dots, x_n)$ such that each x_i is either 0 or m as the "n-dimensional m-by cube." This cube is cut into m^n unit cubes by $(m-1)n$ planes parallel to its faces, with the vertices of the little cubes at lattice points that belong to the m-by cube. These vertices are the points $(x_1, x_2, \dots, x_n)$ such that each x_i is one of the numbers $0, 1, \dots, m$; hence, the total number of such lattice points is $(m+1)^n$.

We now generalize the results of Section 4 by counting the lattice points in the m-by cube that lie on the plane $x_1 + x_2 + \cdots + x_n = k$, $0 \le k \le mn$. This number, denoted $N_k(n, m)$, is merely the number of non-negative integral solutions of $x_1 + x_2 + \cdots + x_n = k$, $0 \le x_i \le m$, and is well known to combinatorialists; it is given by the following formula:

$$N_k(n,m) = \sum_j (-1)^j \binom{n}{j} \binom{k+n-1-j(m+1)}{n-1}, \tag{1}$$

where the sum is over $j = 0, 1, \dots, [k/(m+1)]$, and as usual $[x]$ is the integer part of x. This formula's derivation appears in many books on combinatorics (for example, Riordan [18, p. 104] or Vilenkin [22, pp. 98–100]). Since the coefficient of t^k in the expansion of $(1 + t + t^2 + \cdots + t^m)^n$ is precisely the number of ways that k can be represented as an ordered sum of the integers $x_1, x_2, \dots, x_n$ with $0 \le x_i \le m$, we see that this coefficient is $N_k(n, m)$. Therefore,

$$\frac{\left(1-t^{m+1}\right)^n}{(1-t)^n} = \left(1 + t + t^2 + \cdots + t^m\right)^n = \sum_{k=0}^{\infty} N_k(n,m) t^k. \tag{2}$$

We leave it as reader recreation to use this to derive formula (1). *Hint*: Multiply the binomial expansion of $(1 - t^{m+1})^n$ by

$$\frac{1}{(1-t)^n} = \sum_{j=0}^{\infty} \binom{n+j-1}{n-1} t^j. \tag{3}$$

The case $m = 1$ corresponds to slices of the unit n-cube, and here we already know that $N_k(n, 1) = \binom{n}{k}$. Thus, we have the *none too obvious* relationship

$$\binom{n}{k} = \sum_{j=0}^{[k/2]} (-1)^j \binom{n}{j}\binom{k+n-2j-1}{n-1}. \tag{4}$$

Let us find the number of lattice points in the 4-dimensional 3-by cube that lie on the plane $x_1 + x_2 + x_3 + x_4 = 5$. This corresponds to $n = 4$, $m = 3$, and $k = 5$, so the number we seek is

$$N_5(4,3) = \sum_{j=0}^{1} (-1)^j \binom{4}{j}\binom{8-4j}{3} = \binom{8}{3} - \binom{4}{1}\binom{4}{3} = 56 - 16 = 40.$$

We can also track down this number in the analogue of Pascal's triangle corresponding to $(1 + t + t^2 + t^3)^n$ shown in FIGURE 12. Note that here to get each entry in the numerical triangle we take the sum of the *four* nearest numbers above it.

Triangle based on $(1 + t + t^2 + t^3)^n$

$n = 0$:							1						
$n = 1$:						1	1	1	1				
$n = 2$:					1	2	3	4	3	2	1		
$n = 3$:			1	3	6	10	12	12	10	6	3	1	
$n = 4$:	1	4	10	20	31	40	44	40	31	20	10	4	1
	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$							

FIGURE 12

Let us turn to the geometric analogue of this calculation in terms of slices of the 4-dimensional 3-by cube. FIGURE 13 displays the first four rows of the pictorial triangle of slices of 3-by cubes of dimensions $n = 0, 1, 2, 3$. At the bottom of the figure we have the slice of the 4-dimensional 3-by cube by the plane $x_1 + x_2 + x_3 + x_4 = 5$, showing how it is obtained as the convex hull of appropriately positioned copies of the four nearest figures above it.

Justification of the generating rule for this pictorial triangle is as in §2. The lattice points of the $(n + 1)$-dimensional 3-by cube fall into four layers corresponding to points with last coordinate 0, 1, 2, or 3. The slicing hyperplane that yields the figure in position k in the row corresponding to dimension $n + 1$ intersects the four layers in the figures that appear in positions $k - 3, k - 2, k - 1$, and k of the row corresponding to dimension n.

The pictorial representation extends to all m-by cubes in all dimensions n. For general m and n the generating rule is given by the recursion relation

$$N_k(n+1,m) = \sum_{j=0}^{m} N_{k-j}(n,m), \tag{5}$$

which can be seen either geometrically as above, or algebraically using the fact that

$$(1+t+t^2+\cdots+t^m)^{n+1}=(1+t+t^2+\cdots+t^m)(1+t+t^2+\cdots+t^m)^n.$$

One last observation about m-by cubes: We could have used the generating function

$$(1+w+w^2+w^3)(1+x+x^2+x^3)(1+y+y^2+y^3)(1+z+z^2+z^3),$$

for example, as a labeling device for lattice points in the 4-dimensional 3-by cube, with $w^2xz^2=w^2x^1y^0z^2$, say, corresponding to the point (2, 1, 0, 2). The 40 lattice points belonging to the amputated regular tetrahedron at the bottom of FIGURE 13 correspond to certain monomials of degree 5 in the product. Those on the bottom layer correspond to the last coordinate 0, so the chosen monomials contain only w, x, y, with z to the 0-th power. Similarly, the points in the next layer up have z appearing to the first power, and so on for the other two layers. FIGURE 14 shows the various layers with the w, x, y parts of the corresponding monomials.

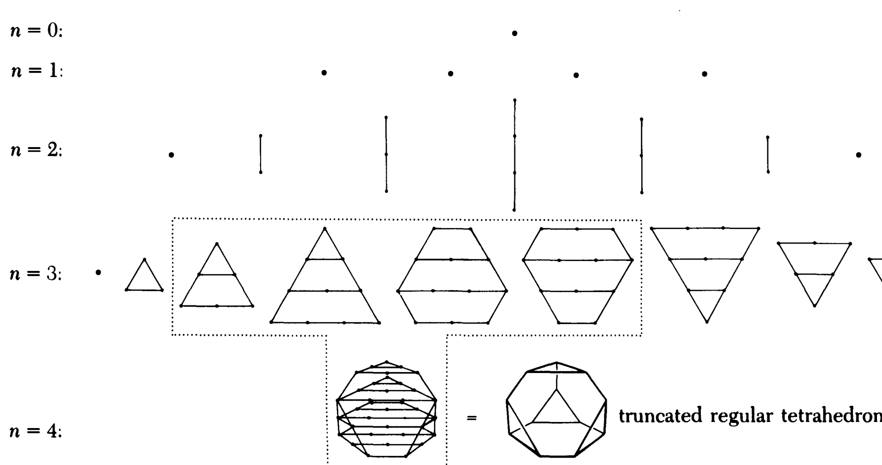


FIGURE 13

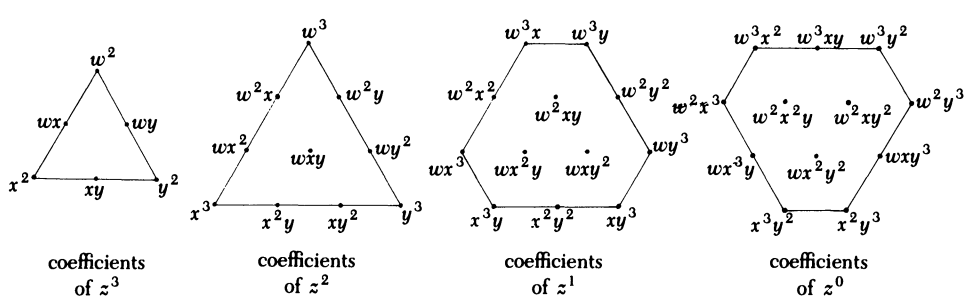


FIGURE 14

6. Areas of slices

The results of Section 5 give us a means of calculating the $(n-1)$-dimensional volume (alias "area") of any slice of the unit n-cube by a plane perpendicular to the main diagonal. Our method of counting lattice points on a slice and then taking a limit is essentially that used by Pólya [14] in calculating the volume of a truncated n-cube. The formulas we derive in this section have a variety of applications to probability (some of which we consider later) and have a history going back to Laplace [10]. Pólya's paper [14] contains references to several authors who have dealt with these questions, including the physicist Arnold Sommerfeld. We shall say a bit more about these matters at the end of this article.

Imagine the unit n-cube sliced by the plane with equation $x_1 + x_2 + \cdots + x_n = t$, $0 \le t \le n$; the $(n-1)$-dimensional volume of this slice we shall refer to as the "area" of the slice and denote by $A(t)$. To get a formula for $A(t)$, we subdivide the cube into $1/m$ by $1/m$ by...by $1/m$ congruent cubes, for m large, and, assuming t is such that the plane contains subdivision points, use the number of lattice points on the hyperplane, multiplied by an appropriate factor, as an approximation for $A(t)$.

Let C be the standard unit n-cube, and partition C into m^n congruent little cubes by planes parallel to the faces of C. (We may imagine subdividing a large m-by cube, as in Section 5, and then shrinking by a factor of m.) The vertices of the little cubes have their coordinates among the numbers $0, 1/m, 2/m, \dots, (m-1)/m, 1$; we shall refer to these points as "subdivision points."

Now let H be the slicing plane determined by $x_1 + x_2 + \cdots + x_n = t$, with $t = p/q$, where p and q are non-negative integers. If d is the distance from the origin to the point where H intersects the main diagonal of C, and D is the length of this diagonal ($D = \sqrt{n}$), we wish to measure $r = d/D = t/n = p/qn$. We then have that $d = r\sqrt{n} = t/\sqrt{n}$.

To count the number of subdivision points on H, we magnify C by the factor m, obtaining the standard m-by cube, and count the number of lattice points that belong to the m-cube and lie on the image of H under magnification, namely on the plane $x_1 + x_2 + \cdots + x_n = mt$. So that we can use formula (1), which requires $k = mt$ be an integer, we assume $m = Mq$, for some integer M. Then $mt = Mqt = Mp$. Thus, the number of lattice points we seek is $N_k(n, m)$, with $k = Mp$ and $m = Mq$, and this is precisely the number of subdivision points on $H \cap C$. More conveniently denoting this number by N, we have from (1) that

$$N = \sum_{j=0}^{[k/(m+1)]} (-1)^j \binom{n}{j} \binom{k+n-j(m+1)-1}{n-1}. \tag{6}$$

Since $k = mt = rmn$, we have (after some cancellation of factorials)

$$\binom{k+n-j(m+1)-1}{n-1} = \frac{1}{(n-1)!}((rn-j)m-j+(n-1)) \times((rn-j)m-j+(n-2)) \cdots ((rn-j)m-j+(1)). \tag{7}$$

We expand the last product in powers of m and obtain a polynomial of degree $n-1$ in m, with leading coefficient $(rn-j)^{n-1}/(n-1)!$. Thus, (6) takes the form

$$N = \frac{m^{n-1}}{(n-1)!} \sum_{j=0}^{[k/(m+1)]} (-1)^j \binom{n}{j} (rn-j)^{n-1} + \cdots, \tag{8}$$

where the later terms are multiplied by powers of m lower than $n-1$.

It can be shown that the plane H is tiled by congruent parallelepipeds having vertices at points $(x_1, x_2, \dots, x_n)$ with $x_i = 0, 1/m, 2/m, \dots, (m-1)/m$, or 1, and each parallelepiped having $(n-1)$-dimensional volume $\sqrt{n}/m^{n-1}$. Thus, $N\sqrt{n}/m^{n-1}$

is a good approximation of the area $A(t)$ if m is large. Using (8) we have

$$\frac{N\sqrt{n}}{m^{n-1}} = \frac{\sqrt{n}}{(n-1)!}\sum_{j=0}^{[k/(m+1)]}(-1)^j\binom{n}{j}(rn-j)^{n-1} + \cdots, \tag{9}$$

where the later terms are multiplied by powers of $1/m$ greater than or equal to 1. We now let $M \to \infty$, with $m = Mq$ and $rn = t = p/q$ fixed. The lefthand side tends to $A(t)$, while on the righthand side all terms except the first tend to 0. Since $[k/(m+1)] \to [t]$ as $M \to \infty$, and $k = Mp$ and $m = Mq$, we obtain

$$A(t) = \frac{\sqrt{n}}{(n-1)!}\sum_{j=0}^{[t]}(-1)^j\binom{n}{j}(t-j)^{n-1}, \tag{10}$$

where $t = rn$ is a rational number. Of course, it follows that (10) holds for all t satisfying $0 \le t \le n$. Another way to interpret the sum in (10) is to observe that we add over $j = 0, 1, 2, \dots,$ stopping when $t - j$ becomes negative.

For those who have taken to heart our injunction that the essence of mathematics is suspicion and who lack faith in our derivation of (10), we do *not* offer an alternative proof, but instead employ a much more powerful method for instilling conviction: We present some *examples*.

Let us begin by calculating the 3-dimensional volume of a central slice of the unit 4-cube. For the slicing plane with $x_1 + x_2 + x_3 + x_4 = t$, $r = t/4$ represents the proportion of the diagonal cut off by the plane. So for a central slice, $r = \frac{1}{2}$, and $t = 2$. Then (10) gives

$$A(2) = \frac{1}{3}\sum_{j=0}^{2}(-1)^j\binom{4}{j}(2-j)^3 = \frac{1}{3}(2^3 - 4\cdot 1^3) = \frac{4}{3}. \tag{11}$$

Furthermore, recall from Section 3 that a central slice of the unit 4-cube is a regular octahedron of edgelength $\sqrt{2}$; it is the intersection of two oppositely oriented regular tetrahedra of edgelength $2\sqrt{2}$, as indicated in FIGURE 15.

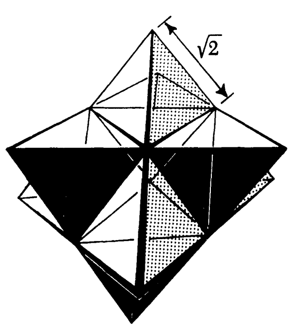


FIGURE 15

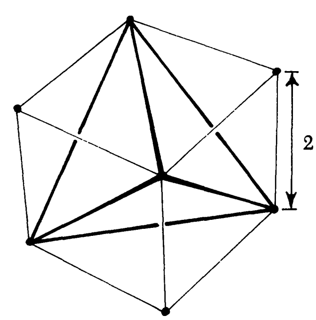


FIGURE 16

Inasmuch as the octahedron is the result of amputating from either of the large tetrahedra four little tetrahedra of half the edgelength, the volume of the octahedron is

$$T - 4\cdot\frac{T}{8} = \frac{T}{2}, \tag{12}$$

where T is the volume of the tetrahedron. But a regular tetrahedron of edgelength $2\sqrt{2}$ sits comfortably in a cube of edgelength 2, shown in FIGURE 16, as a consequence of which we may verify that $T = \frac{8}{3}$, so that from (12) we get $\frac{4}{3}$ for the volume of the octahedron, in confirmation of result (11).

The difference in (12) is identical to the difference in (11) with $T = \frac{8}{3}$. In fact, for $1 \leq t \leq 2$ the sum in (10) takes the form

$$\frac{\sqrt{n}}{(n-1)!}\left(t^{n-1} - n(t-1)^{n-1}\right), \tag{13}$$

representing the volume of an $(n-1)$-dimensional regular simplex of edgelength $t\sqrt{2}$ minus the volumes of n simplices of edgelength $(t-1)\sqrt{2}$, based on the fact that the $(n-1)$-dimensional volume of a regular $(n-1)$-dimensional simplex of edgelength s is [4, p. 295]

$$\sqrt{\frac{n}{2^{n-1}}}\,\frac{s^{n-1}}{(n-1)!}. \tag{14}$$

As we know from Section 3, the slices of the unit n-cube are amputated regular simplices. When the cut-off portions overlap, we cannot compute the volume by a single subtraction as in (13), but instead must use the "inclusion-exclusion principle," which is precisely what formula (10) exhibits. It represents the volume of a large simplex, minus the volumes of n amputated simplices, plus the $\binom{n}{2}$ volumes of simplices formed by overlaps corresponding to edges, minus the $\binom{n}{3}$ volumes of simplices formed by overlaps corresponding to 2-dimensional faces, and so forth. This principle can be used to give another proof of (10) and is discussed in Pólya [14] in a slightly different form.

As another application of (10) we calculate the 3-dimensional volume of a slice of the unit 4-cube by a plane perpendicular to the diagonal and $r = \frac{5}{12}$ of the way from the origin to the opposite vertex. This is a slice by the plane with $x_1 + x_2 + x_3 + x_4 = t = 4r = \frac{5}{3}$. Thus, (10) yields

$$A\left(\frac{5}{3}\right) = \frac{1}{3}\sum_{j=0}^{1}(-1)^j\binom{4}{j}\left(\frac{5}{3}-j\right)^3 = \frac{1}{3}\left(\left(\frac{5}{3}\right)^3 - 4\left(\frac{2}{3}\right)^3\right) = \frac{31}{27}. \tag{15}$$

To reassure ourselves with independent verification, we note that this slice is similar to the amputated regular tetrahedron at the bottom of FIGURE 13 and has one-third of its edgelength. For further reader recreation, check that the required volume is that of a regular tetrahedron of edgelength $5\sqrt{2}/3$ with four regular tetrahedra of edgelength $2\sqrt{2}/3$ cut off from its vertices.

7. Volumes of slabs

Certain applications to probability problems require calculation of the volume of a "slab" of a cube, i.e., the volume of the portion between two planes perpendicular to a main diagonal. This can be found by integrating the cross-sectional area over an appropriate range of values; all we need do is calculate the volume of the part of the cube on one side of a plane and then find the volumes of slabs by subtraction.

Let C be our standard unit cube, and H(t) be the plane with $x_1 + \cdots + x_n = t$, $0 \leq t \leq n$. Since the distance from the origin to H(t) is $t/\sqrt{n}$, the volume between H(t) and H($t + \Delta t$) is approximately

$$\Delta V(t) = \frac{A(t)\,\Delta t}{\sqrt{n}},$$

where $A(t)$ is the cross-sectional area given in (10). Thus, if we let $V(t)$ be the volume of the part of the cube between the origin and H(t), we get via integration

$$V(t)=\frac{1}{\sqrt{n}}\int_0^t A(u)\,du=\frac{1}{n!}\sum_{j=0}^{[t]}(-1)^j\binom{n}{j}(t-j)^n. \tag{16}$$

This formula, which goes back to Laplace [**10**], was obtained directly by Pólya [**14**] counting lattice points and taking a limit.

We pause to consider some special cases. If $0\le t\le 1$, the plane H(t) cuts off a corner of the cube at the origin. The piece cut off is a non-regular n-dimensional simplex with n mutually perpendicular edges of length t emanating from its vertex at the origin (for $n=3$, just a "trirectangular tetrahedron"). This simplex has n-dimensional volume $t^n/n!$, which is exactly what (16) gives for these values of t.

When $t=n/2$, the plane cuts the cube in half, so $V(n/2)=\frac{1}{2}$. If we use this in (16) we obtain

$$\sum_{j=0}^{[n/2]}(-1)^j\binom{n}{j}(n-2j)^n=(2^{n-1})n!, \tag{17}$$

one of the formulas found in Laplace [**10**, p. 171].

The probability problem we consider in the next section requires the volume of a unit cube's slab of a specified width. We need the volume of the standard unit n-cube that lies between the planes H$(k-\frac{1}{2})$ and H$(k+\frac{1}{2})$, where k is a given integer, $0\le k\le n$. Using (16), we find

$$V(k+\tfrac{1}{2})-V(k-\tfrac{1}{2})=\frac{1}{n!}\sum_{j=0}^{k}(-1)^j\binom{n}{j}(k+\tfrac{1}{2}-j)^n$$
$$-\frac{1}{n!}\sum_{j=0}^{k-1}(-1)^j\binom{n}{j}(k-\tfrac{1}{2}-j)^n, \tag{18}$$

which after fiddling with sums, changing index, applying Pascal's identity, and simplifying becomes

$$V(k+\tfrac{1}{2})-V(k-\tfrac{1}{2})=\frac{1}{n!2^n}\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(2k-2j+1)^n. \tag{19}$$

For example, when $n=3$, we find the volumes of the four slabs of the unit cube between the planes H$(-\frac{1}{2})$, H$(\frac{1}{2})$, H$(\frac{3}{2})$, H$(\frac{5}{2})$, H$(\frac{7}{2})$ to be

$$\frac{1}{48},\quad\frac{23}{48},\quad\frac{23}{48},\quad\frac{1}{48}. \tag{20}$$

The slabs in question are indicated in FIGURE 17.

Ignoring the factor $1/n!2^n$, we find the integers obtained from (19) form an interesting numerical triangle. We denote these numbers $S(k,n-k)$, in honor of David Slepian, who rediscovered these volume formulas and some of the associated combinatorics over three decades ago and wrote them up in an unpublished technical memorandum [**19**]. We have from (19)

$$V(k+\tfrac{1}{2})-V(k-\tfrac{1}{2})=\frac{1}{n!2^n}S(k,n-k). \tag{21}$$

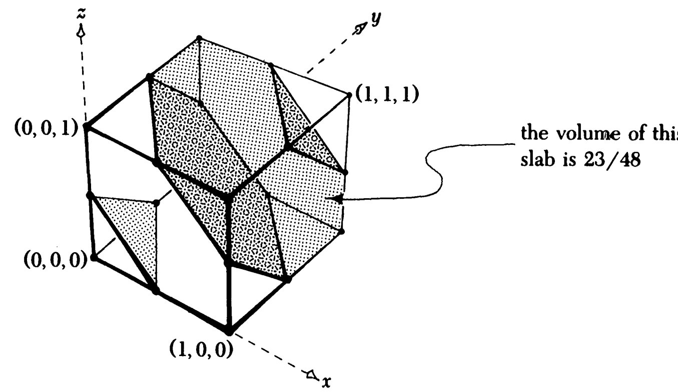


FIGURE 17

Then $S(0,0)=1$; $S(1,0)=S(0,1)=1$; $S(2,0)=S(0,2)=1$, and $S(1,1)=6$, while the numerators in (20) are $S(3,0)$, $S(2,1)$, $S(1,2)$, and $S(0,3)$, respectively. We can arrange these numbers in a triangle as in FIGURE 18, with each row corresponding to a fixed value of n.

The arrows and little numbers in the array indicate the generating rule for the triangle of the $S(k,n-k)$. For example, to get $S(3,1)=S(3,4-3)=76$, we take $3(23)+7(1)$, using the nearest pair in the preceding row. This Pascalish generating rule is expressed as

$$S(k,n-k)=(2n-2k+1)S(k-1,n-k)+(2k+1)S(k,n-k-1). \quad (22)$$

Slepian also considered an associated numerical triangle of numbers $R(k,n-k)$ defined by

$$R(k,n-k)=\sum_{j=0}^{k}(-1)^{j}\binom{n}{j}(k-j)^{n-1}. \quad (23)$$

In view of (10), $\sqrt{n}\,R(k,n-k)/(n-1)!$ is the area of the slice of the unit n-cube by the plane $x_1+\cdots+x_n=k$. We could use (16) to verify that $R(k+1,n-k)/n!$ is also the volume of that part of the unit n-cube between the planes with $x_1+\cdots+x_n=k$ and $x_1+\cdots+x_n=k+1$. We have $R(1,1)=1$; $R(1,2)=R(2,1)=1$; $R(3,1)=R(1,3)=1$, and $R(2,2)=4$. FIGURE 19 shows the numerical triangle of these numbers.

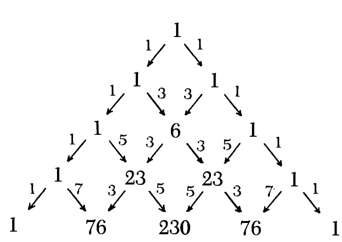


FIGURE 18

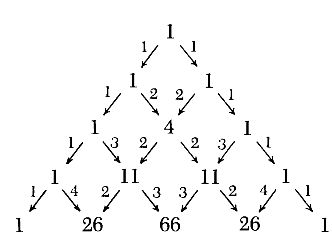


FIGURE 19

Again, the arrows and little numbers give the generating rule for the array. For example, $R(2,4)=26=4(1)+2(11)=4R(1,4)+2R(2,3)$, and the general rule is

$$R(k,n-k)=(n-k)R(k-1,n-k)+kR(k,n-k-1). \quad (24)$$

Note that the nth row in FIGURE 19 corresponds to dimension n, and k ranges from 1 to n.

Experts in combinatorics will recognize (as did Slepian) that the numbers in FIGURE 19 are the "Eulerian numbers." (See Sloane **[20]** or Riordan **[18]**.) The fact that the sum of the entries in the nth row is $n!$ corresponds geometrically to the fact that the entries divided by $n!$ give the volumes of successive slabs of the unit n-cube, and the sum of these volumes is 1. A modest variety of secrets in the Eulerian triangle is uncovered in Logothetti **[13]**.

Compulsive readers may use (10) to verify Slepian's observation that the central slice of a unit n-cube has $(n-1)$-dimensional volume $\frac{\sqrt{n}}{(n-1)!}R(n/2, n/2)$, if n is even, and $\frac{\sqrt{n}}{2^{n-1}(n-1)!}S((n-1)/2, (n-1)/2)$, if n is odd.

8. A Putnam problem

The 1976 William Lowell Putnam Competition problem B-5 requests evaluation of $\sum_{j=0}^{n}(-1)^j\binom{n}{j}(t-j)^n$. The resemblance of this sum to the sum in (16) suggests we take the high road, a geometric approach.

Since the two planes with equations $x_1 + \cdots + x_n = t$ and $x_1 + \cdots + x_n = n - t$ are equidistant from the center of the unit n-cube, we have for the truncated volumes $V(t) + V(n-t) = $ (volume of unit cube) $= 1$. Using formula (16) we get

$$\sum_{j=0}^{[t]}(-1)^j\binom{n}{j}(t-j)^n + \sum_{j=0}^{[n-t]}(-1)^j\binom{n}{j}(n-t-j)^n = n!.$$

Changing the second index from j to $n-j$ gives that second sum the same form as the first, but summed from $n-[n-t]$ to n. Worthy readers will then find that the sums combine to give

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(t-j)^n = n!.$$

Since the expression on the left is a polynomial, and this holds for $0 \le t \le n$, we see that the result holds for all t.

9. The probability of round off

Hilton and Pedersen **[9]** give the following example of a less conventional school problem with useful arithmetic implications and a striking geometric solution.

> Suppose numbers x and y are selected randomly from the closed interval $[0, 1]$. Then the integer nearest to $x + y$ must be one of 0, 1, or 2. What are the respective probabilities of these outcomes?

By looking at FIGURE 20, taken from **[9]**, we immediately see that the probability of $x + y$ rounding off to k is the area of the region labeled k.

Therefore, if $P(k)$ is the probability of $x + y$ rounding off to k, we have $P(0) = P(2) = 1/8$ and $P(1) = 6/8$. The fact that the numerators reproduce the row corresponding to $n = 2$ in FIGURE 18 is no coincidence, as a glance at equation (21) and some concerted thought show. This generalizes to more than two numbers:

Let n be fixed, and suppose numbers $x_1, \dots, x_n$ are randomly selected from the closed interval $[0,1]$. Then the probability that $x_1 + \cdots + x_n$ rounds off to k is $S(k, n-k)/n!\,2^n$.

This follows because those $(x_1, \dots, x_n)$ that belong to the unit n-cube and have $x_1 + \cdots + x_n$ rounding off to k are precisely those points of the cube satisfying $k - \frac{1}{2} \le x_1 + \cdots + x_n \le k + \frac{1}{2}$. The ratio of this slab's volume to that of the entire cube is just $V(k+\frac{1}{2}) - V(k - \frac{1}{2})$, as in (21).

The row of numbers (20) gives the probabilities that $x + y + z$ rounds off to 0, 1, 2, or 3, respectively, when x, y, z are randomly chosen from $[0,1]$, and the geometric interpretation is embodied in FIGURE 17.

In the language of probability, the function $V(t)$ in equation (16) is simply the "cumulative distribution function for the sum of n independent random variables uniformly distributed in the interval [0, 1]." This explains the success of probabilistic arguments in connection with the geometric problems of Section 11. The full story is told in Feller [**6**, I.9].

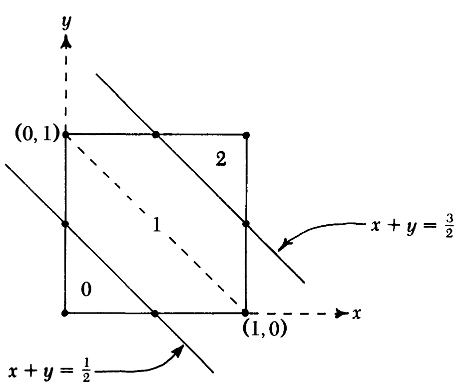


FIGURE 20

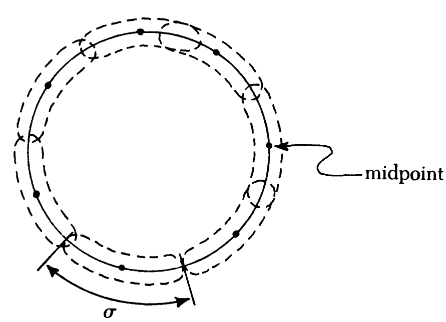


FIGURE 21

10. The probability of a (non-governmental) cover-up

The following geometric probability problem and its generalizations are completely treated in Solomon [**21**, Chap. 4], which also tells the interesting history of the problem. Feller [**6**, p. 28] treats the problem in the context of a discussion of the function in (16) as a distribution function.

Let K be a circle of *circumference* 1, and suppose $n \ge 2$ arcs of length σ are distributed randomly over K. What is the probability that these arcs cover K? (See FIGURE 21.)

If K is covered, the midpoints of the n arcs will subdivide the circumference of K into n arcs of lengths $x_1, \dots, x_n$, with $x_1 + \cdots + x_n = 1$. The circle is covered precisely when $x_i \le \sigma$ for all $i = 1, 2, \dots, n$. Therefore, an equivalent formulation of the problem is the following:

Given $x_1, \dots, x_n$ satisfying $0 \le x_i \le 1$, $i = 1, 2, \dots, n$, and $x_1 + \cdots + x_n = 1$, what is the probability that $x_i \le \sigma$ for all i?

The set of such $(x_1,\dots,x_n)$ is the $(n-1)$-dimensional regular simplex S of edgelength $\sqrt{2}$ in n-dimensional Euclidean space with vertices at $(1,0,0,\dots,0),(0,1,0,\dots,0),\dots,(0,0,0,\dots,1)$. The intersection of S with the standard n-cube C_σ of edgelength σ consists of those $(x_1,\dots,x_n)\in S$ that satisfy $0\le x_i\le\sigma$, for $i=1,2,\dots,n$. Hence, the desired probability is the ratio of the $(n-1)$-dimensional volumes,

$$P(\sigma)=\frac{A(S\cap C_\sigma)}{A(S)}. \tag{25}$$

To calculate $A(S\cap C_\sigma)$, magnify the figure by the factor $1/\sigma$. We then have the standard unit n-cube intersected by the plane with $x_1+\cdots+x_n=1/\sigma$. From (10) we know that the magnified intersection has $(n-1)$-dimensional volume

$$\frac{\sqrt{n}}{(n-1)!}\sum_{j=0}^{[1/\sigma]}(-1)^j\binom{n}{j}\left(\frac{1}{\sigma}-j\right)^{n-1}=\frac{\sqrt{n}}{\sigma^{n-1}(n-1)!}\sum_{j=0}^{[1/\sigma]}(-1)^j\binom{n}{j}(1-\sigma j)^{n-1} \tag{26}$$

Since this was obtained after magnification by $1/\sigma$, the original $(n-1)$-dimensional volume, $A(S\cap C_\sigma)$ is found after division by $(1/\sigma)^{n-1}$. But from (14), with $s=\sqrt{2}$,

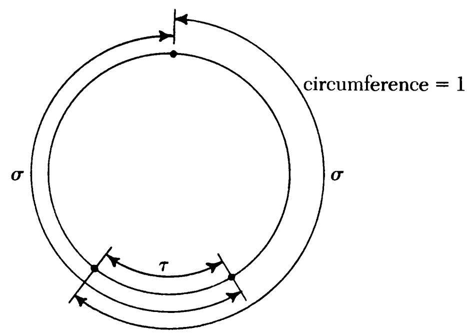


FIGURE 22

we have $A(S)=\sqrt{n}/(n-1)!$. Thus, dividing by these factors in (26), we get the probability we want, *viz.*

$$P(\sigma)=\sum_{j=0}^{[1/\sigma]}(-1)^j\binom{n}{j}(1-\sigma j)^{n-1}. \tag{27}$$

Note that we must have $\sigma>1/n$ in order that the probability of coverage be nonzero, so it is assumed in (27) that $1/\sigma<n$. Geometrically, this corresponds to the assumption that $S\cap C_\sigma$ is more than a point.

Applying (27) in the case $n=2$ gives $P(\sigma)=2\sigma-1$. This is plausible, since we may see with a sketch that if we fix one arc of length σ, then the set of centers of other arcs of length σ that together with the fixed arc give coverage is itself an arc of length $\tau=2\sigma-1$. (See FIGURE 22.)

11. Cube slicing, from the nineteenth century to the present

Laplace [10, p. 170] gave an integral formula that, in our setup, amounts to

$$A(t)=\frac{2\sqrt{n}}{\pi}\int_0^\infty\left(\frac{\sin u}{u}\right)^n\cos((n-2t)u)\,du, \tag{28}$$

where $A(t)$ is the $(n-1)$-dimensional volume of the slice of the unit n-cube as in (10). Pólya [14] proved a general integral formula that gives areas of cube slices by *arbitrary* planes, reducing to (28) when the slice is perpendicular to a main diagonal. Our derivation of (10) is similar in spirit to Pólya's, involving counting lattice points and taking a limit.

The distance from the plane with $x_1 + \cdots + x_n = t$ to the origin is $t/\sqrt{n}$. If $t \geq n/2$, then $s = t/\sqrt{n} - \sqrt{n}/2$ is the distance from the plane to the center of the cube. In terms of this distance, (28) becomes

$$A\left(\frac{n}{2} + s\sqrt{n}\right) = \frac{2\sqrt{n}}{\pi}\int_0^\infty \left(\frac{\sin u}{u}\right)^n \cos(2\sqrt{n}\,su)\,du. \tag{29}$$

Laplace and Pólya both gave proofs that

$$\lim_{n\to\infty} A\left(\frac{n}{2} + s\sqrt{n}\right) = \sqrt{\frac{6}{\pi}}\,e^{-6s^2} \tag{30}$$

The case of a central slice is especially interesting. Here $s = 0$, and we obtain

$$\lim_{n\to\infty} A\left(\frac{n}{2}\right) = \sqrt{\frac{6}{\pi}} \doteq 1.382. \tag{31}$$

Thus, our $(n-1)$-dimensional volume of the central slice of the unit n-cube (perpendicular to a main diagonal) approaches $\sqrt{6/\pi}$ as n approaches ∞. So from (28) we have

$$\lim_{n\to\infty} \frac{2\sqrt{n}}{\pi}\int_0^\infty \left(\frac{\sin u}{u}\right)^n du = \sqrt{\frac{6}{\pi}}\,. \tag{32}$$

Using probabilistic methods, Hensley [8] showed that there exists an upper bound, independent of n, on the $(n-1)$-dimensional volumes of *all* central slices of a unit n-cube (not just those perpendicular to a body diagonal). His conjecture that $\sqrt{2}$ is such an upper bound (and this is best possible, since a central slice of an n-cube that contains an $(n-2)$-dimensional face has $(n-1)$-dimensional volume $\sqrt{2}$) was proved by Ball [1] in 1986. Both Ball and Hensley used probabilistic methods, ending up making ingenious estimates on integrals corresponding to the integral formula for volume treated by Pólya.

In a later paper, Ball [2] observed that this bound on areas of slices provides a remarkably simple solution, at least for dimension $n \geq 10$, to a famous problem of Busemann and Petty: Must an n-dimensional centrally symmetric convex body have greater volume than another such body with the same center if each slice through the center of the first body has greater $(n-1)$-dimensional volume than the corresponding slice through the second? Ball observed that for $n \geq 10$ each central slice of an n-dimensional ball of unit volume has $(n-1)$-dimensional volume greater than $\sqrt{2}$, and hence greater than the corresponding slice of the unit cube. Thus, a slightly smaller ball will still have its slices larger than those of the cube, yet have smaller volume. This lays to rest the cases for $n \geq 10$, but the question is still open in dimensions $3, 4, \ldots, 9$. Larman and Rogers [11] earlier gave a more complicated probabilistic argument that settled the problem for $n \geq 12$.

The formula for the volume of an n-cube truncated by an arbitrary plane was also treated with brevity and elegance by Barrow and Smith [3] using spline notation. They gave a combinatorial version, similar to a formula of Pólya [14], and indicated the probabilistic significance.

We now come to some discussion of central symmetry that, while a little specialized, connects well with the rest of this paper, is difficult to find in the literature, and gives well-deserved publicity to an interesting unsolved problem.

The central slice perpendicular to the main diagonal of the unit n-cube is also of interest in connection with a problem of Fáry and Rédei **[5]**. Recall from Section 3 that such a slice is the intersection of two oppositely oriented regular $(n-1)$-dimensional simplices having the same centroid and edgelength $n/\sqrt{2}$. FIGURE 15 illustrates the case $n=4$. It happens that this intersection (a regular octahedron in FIGURE 15) is the centrally symmetric convex body of largest volume that fits inside the simplex. The ratio of this intersection's volume to that of the simplex tells us, in a sense, how close the simplex is to being centrally symmetric. For example, in FIGURE 23 we see that this "measure of central symmetry" for an equilateral triangle is $\frac{2}{3}$.

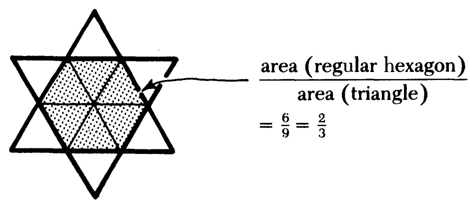


FIGURE 23

As another example, since the regular octahedron in FIGURE 15 has half the volume of the tetrahedron, the tetrahedron's measure of symmetry is $\frac{1}{2}$.

In general, the volume of a regular n-dimensional simplex of edgelength $(n+1)/\sqrt{2}$ is by (14)

$$\sqrt{\frac{n+1}{2^n}}\,\frac{\left(\frac{n+1}{\sqrt{2}}\right)^n}{n!}=\frac{(n+1)^{n+1/2}}{2^n\,n!}, \tag{33}$$

and the largest centrally symmetric convex subset has n-dimensional volume given by (10) with n replaced by $n+1$ and $t=(n+1)/2$:

$$\frac{\sqrt{n+1}}{2^n\,n!}\sum_{j=0}^{[(n+1)/2]}(-1)^j\binom{n+1}{j}(n+1-2j)^n. \tag{34}$$

Therefore, we see that the measure of symmetry of an n-dimensional regular simplex is

$$\frac{1}{(n+1)^n}\sum_{j=0}^{[(n+1)/2]}(-1)^j\binom{n+1}{j}(n+1-2j)^n. \tag{35}$$

This expression was derived by Fáry and Rédei **[5]**. The equivalent integral form, from (28) with n replaced by $n+1$ and $t=(n+1)/2$, is also given there and attributed to Paul Turán.

If we are interested in how the measure of central symmetry for a regular n-simplex behaves when n is large, we can use the fact, equivalent to (32), that $A(n/2)$ approaches $\sqrt{6/\pi}$ as n approaches ∞. Thus, by (33) a simplex's measure of symmetry behaves like

$$\sqrt{\frac{6}{\pi}} \cdot \frac{2^n n!}{(n+1)^{(n+1)/2}}$$

for large n. Using Stirling's approximation for $n! \sim \sqrt{2\pi n}(n/e)^n$ and a little massaging, we get the large-n behavior of a regular n-simplex's measure of symmetry:

$$\sqrt{3} \cdot \left(\frac{2}{e}\right)^{n+1}.$$

Computation addicts may check how well this approximates the exact measure of symmetry given by (35).

For any convex body K, let S be the centrally symmetric convex body of largest volume contained in K. Then the ratio of S's volume to K's volume may be used as a measure of central symmetry of K. The preceding calculations are of some interest since Fáry and Rédei conjectured that an n-dimensional simplex has the smallest measure of symmetry among all n-dimensional convex bodies. In other words, we expect (35) to give a lower bound for the measure of symmetry of any n-dimensional convex body. This is known to be true in case $n = 2$, but the conjecture is still open for $n \geq 3$. Thus, for example, it is not known whether every 3-dimensional convex body contains a centrally symmetric subset with half the volume.

For any reader who may have persisted with us to the end of the article and may be interested in further applications of this type of analysis, we mention a recent paper of Weissbach **[23]**, wherein he used methods similar to those of this section to prove that if C is the usual unit n-cube centered at the origin, and K is a unit "cross-polytope" (generalized regular octahedron) also centered at the origin, then the volume of $K \cap C$ tends to zero as n approaches ∞.

12. Concluding remark

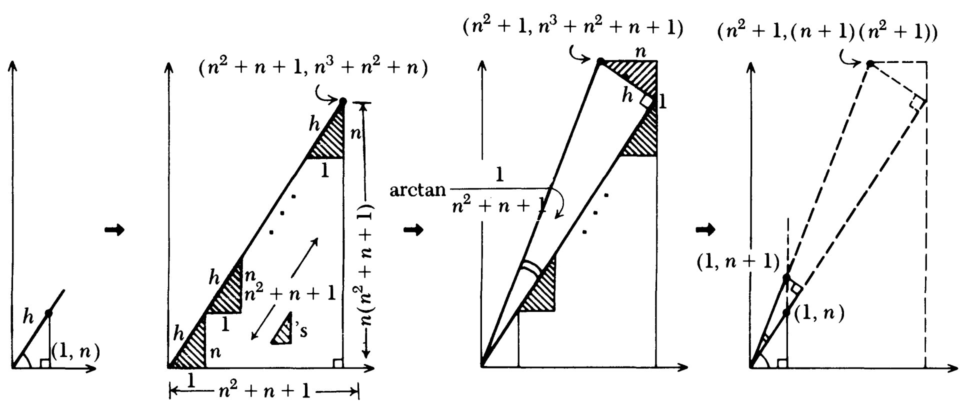
As we finally end our odyssey through cubes and lattice points, amputated simplices, numerical triangles, probabilistic slabs and arcs, and measures of central symmetry, we are gratefully indebted to the referees for their gracious advice and gentle wisdom, which contributed much toward the improvement of this article.

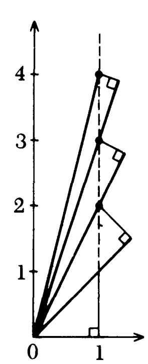
REFERENCES

1. Keith Ball, Cube slicing in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.* 97 (1986), 465–473.
2. Keith Ball, Some remarks on the geometry of convex sets, *Geometric Aspects of Functional Analysis* (1986/87), 224–231. *Lecture Notes in Math.*, 1317, Springer-Verlag, New York, 1988.
3. D. L. Barrow and P. W. Smith, Spline notation applied to a volume problem, *Amer. Math. Monthly* 86 (1979), 50–51.
4. H. S. M. Coxeter, *Regular Polytopes*, 3rd edition, Dover, New York, 1973.
5. I. Fáry and L. Rédei, Der zentralsymmetrische Kern und die zentralsymmetrische Hülle von konvexen Körpern, *Math. Ann.* 122 (1950), 205–220.
6. W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 2, John Wiley and Sons, Inc., New York, 1966.
7. B. Grünbaum, *Convex Polytopes*, Interscience, London, 1967.
8. D. Hensley, Slicing the cube in $\mathbb{R}^n$ and probability (bounds for the measure of a central cube slice in $\mathbb{R}^n$ by probability methods), *Proc. Amer. Math. Soc.* 73 (1979), 95–100.
9. P. Hilton and J. Pedersen, A role for untraditional geometry in the curriculum, *Kolloquium Mathematik-Didaktik der Universität Bayreuth* (December, 1989).
10. P. S. Laplace, *Théorie analytique des probabilités*, Paris, 1812.
11. D. G. Larman and C. A. Rogers, The existence of a centrally symmetric convex body with central sections that are unexpectedly small, *Mathematika* 22 (1975), 164–175.
12. S. Lay, *Convex Sets and their Applications*, John Wiley and Sons, Inc., New York, 1982.

13. D. Logothetti, Rediscovering the Eulerian triangle, *California MathematiCs* 4 (1979), 1, 27–33.
14. G. Pólya, Berechnung eines bestimmten Integrals, *Math. Ann.* 74 (1913), 204–212.
15. G. Pólya, On picture writing, *Amer. Math. Monthly* 63 (1956), 689–697.
16. G. Pólya, *Mathematical Discovery* (combined edition), John Wiley and Sons, Inc., New York, 1981.
17. J. Putz, The Pascal polytope: An extension of Pascal's triangle to N dimensions, *College Math. J.* 17 (1986), 144–155.
18. J. Riordan, *An Introduction to Combinatorial Analysis*, John Wiley and Sons, Inc., New York, 1958.
19. D. Slepian, On the volume of certain polytopes, Technical Memorandum, April 11, 1956, Bell Telephone Laboratories.
20. N. J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, New York, 1973.
21. H. Solomon, *Geometric Probability*, Regional Conference Series in Applied Math. 28, SIAM, 1978.
22. N. Ya. Vilenkin, *Combinatorics*, Trans. by A. Shenitzer and S. Shenitzer, Academic Press, New York, 1971.
23. B. Weissbach, Zu einer Aufgabe von J. M. Wills, *Acta Mathematica Hungarica* 48 (1986), 131–137.

Proof without Words: An Arctangent Identity and Series





$$\arctan n + \arctan\frac{1}{n^2+n+1} = \arctan(n+1)$$

$$\arctan\frac{1}{n^2+n+1} = \arctan(n+1) - \arctan n$$

$$\sum_{n=0}^{\infty}\arctan\frac{1}{n^2+n+1} = \lim_{N\to\infty}\arctan(N+1) = \frac{\pi}{2}$$

—Roger B. Nelsen
Lewis and Clark College
Portland, OR 97219