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Connectivity and Smoke-Rings: Green's Second Identity in Its First Fifty Years

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Introduction

James Clerk Maxwell, in his review of Thomson and Tait's *Treatise on Natural Philosophy*, noted an important innovation in the authors' approach to mathematics:

> The first thing which we observe in the arrangement of the work is the prominence given to kinematics,... and the large space devoted under this heading to what has been hitherto considered part of pure geometry. The theory of curvature of lines and surfaces, for example, has long been recognized as an important branch of geometry, but in treatises on motion it was regarded as lying as much outside of the subject as the four rules of arithmetic or the binomial theorem.
>
> The guiding idea, however... is that geometry itself is part of the science of motion, and that it treats, not of the relations between figures already existing in space, but of the process by which these figures are generated by the motion of a point or a line. [1]

This "guiding idea," which treats geometric entities as physical objects in some sense, had been influential with mathematicians for many years. Countless mathematical problems have their origin in the investigation of the natural world. However, it also happens that the solutions of some problems may be facilitated by attributing physical properties to the mathematical objects under study. In addition, mathematical constructs usually thought of as "purely geometric" may be created by considering such mathematico-physical entities.

It is my purpose in this article to illustrate some aspects of the cross-fertilization of mathematics and physics by examining the development of Green's second identity (known to physicists as Green's theorem) and its generalizations over a fifty-year period, from 1828 to 1878. During this period, despite an increased emphasis on logical rigour in some circles, many mathematicians continued to accept physical proofs of analytic theorems as valid. Such proofs used hypothetical physical properties such as incompressibility to characterize the regions in space; this trend was found most strongly in nineteenth-century British mathematics, though it was not unknown elsewhere.

Green's second identity, well known from vector calculus, states that

Peter Guthrie Tait

$$\iiint (\varphi\nabla^2\psi - \psi\nabla^2\varphi)\,dx\,dy\,dz = \oint\left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right)da.$$

The integration on the left is performed over a region bounded by a closed surface S. The integral on the right is then a surface integral over S, and n is an outward normal to S. Finally, φ and ψ are continuously differentiable real-valued functions (scalar fields) on R^3. This theorem first appeared in a paper by George Green published in 1828, along with a number of other lemmas which Green employed in his study of electrostatics and magnetism. Green's results remained virtually unknown, however, until William Thomson (later Lord Kelvin) obtained two copies of Green's pamphlet in 1845. Green's results subsequently became widely known, and were central to the mathematical theory of potential, one of the most important tools of mathematical physics in the following decades.

Potential theory had originated as a body of results which arose in connection with the efforts of French mathematical physicists (notably Poisson, Laplace, and Biot), to

William Thomson, later Lord Kelvin

extend the methods of Newton. Laplace attempted to explain many natural phenomena as the result of forces proportional to the inverse square of the distance between the interacting objects. To achieve this, it was necessary to determine the integrals of vector forces. Laplace showed that such forces could be treated as what we now term the gradient of a scalar function, and hence was able to simplify the calculations greatly. Such a function, the gradient of which is a force, is known as a potential for that force. (We will also see velocity potentials in the course of this article, which are functions the gradient of which gives a velocity.) Like that of Laplace, Green's work was a contribution both to mathematical physics and to potential theory, since it expresses relationships between potentials and their integrals as well as applying the results to physical problems [2].

At its beginning, the idea of potential was a mathematical convenience. However, by the 1850s it had acquired physical interpretations. In particular, if a vector function has a potential, the integral of that potential along a curve depends only on the endpoints of the integration, and integrals around closed paths are zero. This expresses the fact that the vector function is an exact differential. Physically, this implies that the force described by the function is conservative, so that potential functions are closely associated with potential energy.

Green's 1828 Essay

Green's paper, which was published privately in Nottingham in 1828, was called *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* [3]. George Green (1793–1841), a miller's son, had been given access to the library of a local aristocrat interested in science. This opportunity, and Green's ability, permitted him to master basic works by Laplace, Lagrange, and Poisson. Inspired by Laplace's work on gravitation and Poisson's on electrostatics and magnetism, Green set forth to investigate electrostatics using similar hypotheses but new methods.

Of particular interest to us are Green's general mathematical theorems, presented at the beginning of the paper, which he later applied to particular electrical and magnetic calculations. Green's second identity is the key theorem in this section. It is the essential tool in solving the Laplace equation and the Poisson equation by the method which Green introduced. A detailed discussion of this method, today known as the method of Green's functions, would take us too far afield.

In modern notation, the identity Green proved was

$$\int U\nabla^2 V d^3x + \oint U\frac{\partial V}{\partial n}\,d\sigma = \int V\nabla^2 U d^3x + \oint V\frac{\partial U}{\partial n}\,d\sigma. \tag{1}$$

Here U and V are any two functions which are continuously differentiable in the region of differentiation, and n is now the inward normal from the surface σ. Green used the symbol δ to express what we have denoted by ∇^2. Green's proof of this identity rests on applying integration by parts to the expression

$$\iiint\left\{\frac{\partial V}{\partial x}\cdot\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\cdot\frac{\partial U}{\partial y}+\frac{\partial V}{\partial z}\cdot\frac{\partial U}{\partial z}\right\}dx\,dy\,dz = \int(\nabla V)\cdot(\nabla U). \tag{2}$$

Assuming U, V are sufficiently differentiable, we can integrate by parts in each variable. For example let

$$u = \frac{\partial U}{\partial x}\,dx \quad \text{and} \quad v = \frac{\partial V}{\partial x}\,dx.$$

Then substitution in (2) yields

$$\begin{aligned}&\iint dy\,dz\left(\int\frac{dV}{dx}\frac{dU}{dx}\,dx\right)\\ &\quad=\iint V(x_1)\frac{dU}{dx}\bigg|_{x=x_1}dy\,dz-\iint V(x_0)\frac{dU}{dx}\bigg|_{x=x_0}dy\,dz-\int V\frac{\partial^2 U}{\partial x^2}\,dx\,dy\,dz.\end{aligned}$$

Green then argued that, if σ is a surface element and n an inward normal, we have

$$\iint dy\,dz\left(V(x_1)\frac{dU}{dx}\bigg|_{x_1} - V(x_0)\frac{dU}{dx}\bigg|_{x_0}\right) = -\int d\sigma \frac{\partial x}{\partial n} V\frac{dU}{dx}.$$

Hence the partial integral becomes

$$\iint dy\,dz\left(\int \frac{\partial V}{\partial x}\frac{\partial U}{\partial x}\,dx\right) = -\int d\sigma \frac{\partial x}{\partial n} V \frac{\partial U}{\partial x} - \int V\frac{\partial^2 U}{\partial x^2}.$$

Consequently the result of integration with respect to all three variables gives

$$\iint dx\,dy\,dz(\nabla V)\cdot(\nabla U) = -\int d\sigma V\frac{\partial U}{\partial n} - \int V\nabla^2 U. \tag{3}$$

This is often known as Green's first identity. By symmetry, we may interchange U and V in (3) to obtain the second identity:

$$\int d\sigma V\frac{\partial U}{\partial n} - \int V\nabla^2 U = -\int d\sigma\, U\frac{\partial V}{\partial n} - \int V\nabla^2 U.$$

Many present-day niceties in the proof of this identity were not considered by Green. A full proof involves dealing properly with the relationship between the infinitesimals and the finite, and we must use the equivalence of multiple and interated integrals, which Green did not distinguish. The advances in rigorous analysis due to Cauchy may well have been unknown to Green at this time, since he mentions his limited access to the latest work. Instead his arguments rely on the geometry of infinitesimals. In this, his work resembles that of most of his contemporaries, even in France.

Green's work went almost entirely unnoticed for many years. None of the private subscribers who purchased the pamphlet appears to have been capable of appreciating its worth, and his results and methods remained little known [4]. Green's work might have been forgotten had it not been mentioned by the Irish electrician Robert Murphy. Murphy referred to Green as the originator of the term potential, though Murphy's own definition of potential was erroneous, indicating that he had not actually seen Green's work [5].

Green himself did not revive interest in his earlier work. His efforts in the interim were devoted to further research, and to an education at Cambridge. His other papers met a happier immediate reception; several were published in the *Transactions of the Cambridge Philosophical Society*, where they attracted the interest of the British scientific community. Green thus made a name for himself before his death in 1841, though his reputation was considerably enhanced by the rediscovery of the 1828 *Essay*.

Thomson Rediscovers Green

It was William Thomson (later Lord Kelvin) who first drew the attention of the international scientific world to Green's results. Sometime in 1842 Thomson had read a reference by Murphy to Green's paper; his interest was piqued for several reasons. Thomson was himself then engaged in research on the theory of attraction, and published papers on the subject in 1842 and 1843. Murphy had referred to Green's use of the term potential, a notion which, as Thomson states, was also employed by Gauss with great success in his 1839 paper on inverse-square forces. Thomson doubtless wondered how Green, whose name he knew well, had employed the notion of potential, and was curious about the exact nature of his results.

Thomson was unable to see a copy of Green's work until January 25, 1845, shortly before he was about to embark on a trip to France following the completion of his studies at Cambridge. By chance, Thomson's tutor, Hopkins, had two copies which he had apparently never examined, and sent them with Thomson. Thomson was very impressed with the generality of Green's results, and was soon endeavouring to apply them in his own research. On his arrival in France, Thomson showed the paper to Liouville, Sturm, and Chasles, among others. Soon the Paris mathematical community was well aware of Green's work: for example, Liouville gave Green and Gauss equal credit for the introduction of the term potential in an 1847 paper [**6**]. Thomson sent the other copy of Green's paper to Germany with Cayley, who delivered it to August Crelle, editor of the *Journal für die reine und angewandte Mathematik*. Crelle published a translation of Green's paper in three installments between 1850 and 1854, hence it became well known to interested researchers in Germany. Thus, 25 years after Green's original publication, his methods began to find their way into the scientific literature and textbooks of Europe [**7**]. One of the first to make use of Green's work was Bernhard Riemann (1826–1866), who was then writing his doctoral dissertation at Göttingen.

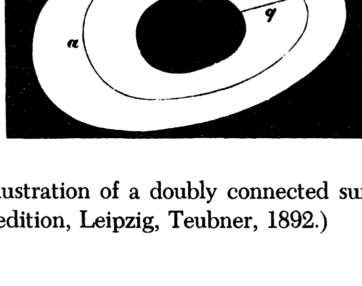
Riemann and Multiply-Connected Regions

Riemann's principal interest in Green's work was in the method of Green's functions, which Green had used to solve boundary-value problems involving functions satisfying Laplace's equation

$$\nabla^2\varphi = 0.$$

Here φ is to be interpreted as the potential function of the electrostatic force due to a charge density on a conductor. Riemann, however, noticed that Green's methods could be useful in the study of functions of a complex variable, since the real and complex parts of such functions must satisfy Laplace's equation. Employing this insight, Riemann developed methods that enabled him to specify a complex function by its boundary values and discontinuities. In so doing, Riemann presented the idea of *multiply-connected regions* of the plane: a region is simply connected if a cross-cut divides it in two, and has connectivity equal to the number of cuts taken to separate it. (See FIGURE 1.) This notion was published in Riemann's dissertation (1851) and found wider circulation with the appearance of his paper on abelian integrals (1857) [**8**]. It was here that it was seen by Hermann von Helmholtz, who was attempting to employ Green's ideas in a different way.

Zweifach zusammenhängende Fläche.



Sie wird durch jeden sie nicht zerstückelnden Querschnitt *q* in eine einfach zusammenhängende zerschnitten. Mit Zuziehung der Curve *a* kann in ihr jede geschlossene Curve die ganze Begrenzung eines Theils der Fläche bilden.

FIGURE 1.
Riemann's illustration of a doubly connected surface. (From B. Riemann, *Gesammelte Math. Werke*, 2nd edition, Leipzig, Teubner, 1892.)

Helmholtz and Vortices

Hermann von Helmholtz's 1858 paper *On Integrals of Hydrodynamic Equations Which Yield Vortex Motion* was also deeply influenced by Green's work [9]. Helmholtz (1821–1894) had become interested in the solution of boundary-value problems in fluid mechanics in connection with his investigation of the physiology of the ear. (He was at that time a professor of anatomy in Bonn [**10**].) Furthermore, Helmholtz saw a parallel between certain problems in hydrodynamics and problems in electromagnetic theory, a longstanding interest of his. Attempts to provide a detailed theoretical treatment of the analogy between electromagnetic theory and fluid dynamics may have been sparked by the superficial resemblance between electrical and hydrodynamical phenomena. The electric current was widely viewed in the mid-nineteenth century as the flow of one or two "electric fluids" along a conductor. The motion of this fluid produces a magnetic effect. André-Marie Ampère demonstrated in the 1820s that magnetism may be explained as the result of hypothetical microscopic electric currents in a body, and hence the existence of the electromagnetic phenomenon should mean that a current gives rise to other currents. If the original current flows in a straight line, the currents responsible for magnetic effects must be helical, forming microscopic vortices.

The researches of Helmholtz and others in this area aimed to make this rather vague picture precise. In the 1858 paper, Helmholtz examined the following question: suppose we are given a closed container filled with a frictionless incompressible fluid. How does action on the boundary of the container affect the motion inside?

Helmholtz apparently saw the value of Green's theorems in such an investigation soon after reading Green's paper, but was kept from working out his ideas because of other academic obligations. However, Helmholtz had also recently read Riemann's paper of 1857, which made it clear to him that Green's theorem could only be used when the regions involved were simply connected. This is because functions with potentials—what we would now term conservative vector fields—may in fact be multiple-valued in multiply-connected regions.

Let us discuss how Helmholtz used Green's theorem. He began with Euler's equation of fluid dynamics, which we may write in vector notation as

$$\vec{F}=\frac{1}{\rho}\nabla\vec{p}+\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)\vec{v}, \tag{4}$$

$$\nabla\cdot\vec{v}=0. \tag{5}$$

Here ρ is the density of the fluid, $\vec{p}$ the pressure, $\vec{v}$ the velocity. Helmholtz supposed that

$$\vec{F}=\nabla V \quad \text{(where } V \text{ is a force-potential)}$$

and

$$\vec{v}=\nabla\varphi \quad (\varphi \text{ is a velocity-potential}).$$

From (5) we have that φ satisfies Laplace's equation, since $\nabla\cdot\vec{v}=\nabla^2\varphi=0$. Helmholtz then noted that this implies

$$\nabla\times\vec{v}=\nabla\times(\nabla\varphi)=0.$$

Bernhard Riemann

If the walls of the container are rigid, this means that the component of velocity perpendicular to the boundary is zero. Hence, if n is an outward normal, $\partial\varphi/\partial n$ is equal to zero everywhere. But by Green's first identity,

$$\int_R (\nabla U \cdot \nabla V)\, dx^3 = \iint V \frac{\partial U}{\partial n}\, ds - \int_R V \nabla^2 U\, dx^3.$$

If $U = V = \varphi$ we have

$$\int_R (\nabla\varphi)^2\, dx^3 = \iint \varphi \frac{\partial \varphi}{\partial n}\, ds = 0, \qquad (\text{Remember } \nabla^2\varphi = 0.)$$

so that $\nabla\varphi = 0$, i.e., no motion is induced in the fluid. Hence, any motion of a fluid in a closed vessel (with simply-connected interior) which has a velocity potential must depend exclusively on a motion of the boundary. Helmholtz went on to show that a motion of the boundary uniquely determines such a motion in the fluid. A further important conclusion stated that vortices can only be produced by a motion which has no velocity potential. More important still, if vortices do exist initially, they are stable

Hermann von Helmholtz

under the action of conservative forces. Either they must be closed tubes, or else they extend from the boundary to the boundary.

When vortices do exist one can consider the portion of fluid without vortices as a multiply-connected region, the vortices being the "holes." Thus to solve boundary value problems in such a region, one would ideally have an extension of Green's theorem to deal with such cases. Helmholtz stressed the desirability of such a generalization, and noted that Riemann's notion of connectivity could readily be extended to three dimensions for this purpose.

In Helmholtz's work the geometric entities immediately become physical. For one thing, they are three dimensional. Also the points of space become associated with the molecules of a fluid, and holes in the space correspond to vortices. In this instance, the geometric entities may be given clear physical interpretation, and physical questions (the solution of specific boundary-value problems, for example) dictate the mathematical problems which are important.

Helmholtz's research was received with greater sympathy by British mathematical physicists, especially Thomson, than by his German colleagues. This occurred in part because of the shared interests of Helmholtz and Thomson in hydrodynamic models for electromagnetic theory, an interest that arose because of their attitude toward the then-prevailing thought on electromagnetic theory in Germany. This theory, based on work by Gauss's collaborator Wilhelm Weber, explained electrical phenomena on the basis of a velocity-dependent force law. Both Helmholtz and Thomson felt that such a force could not satisfy the energy conservation principle. Helmholtz apparently felt as well that his mathematical skills were dimly regarded by his German contemporaries, because he had not been formally trained as a mathematician. Thus it was among British mathematical physicists that Helmholtz's papers were read with greatest interest and understanding.

Tait, Thomson, Smoke-Rings and Atoms

Helmholtz's approach found an enthusiastic admirer in Peter Guthrie Tait (1837–1901), a Cambridge-educated Scot who was teaching in Belfast in 1858. Tait was attempting at that time to master William Rowan Hamilton's method of quaternions, and to demonstrate the physical usefulness of the method by obtaining significant applications. In this respect Helmholtz's work interested him, and he made an English translation for his own use [11].

A parenthetic note about quaternions: Nowadays, this set of objects is most likely to show up in algebra courses or proofs in algebraic number theory which can make use of its properties as a noncommutative division ring. This is quite remote from their original intended use in geometry and analysis. Hamilton invented quaternions in 1843, and introduced with them the idea of operators. Particularly important was the del or nabla operator, our ∇. For Hamilton and Tait, a quaternion described a quotient of what we would term vectors; such a quotient consists of a 4-tuple which describes the stretch and the three rotations which bring an arbitrary pair of vectors into coincidence [12]. Later on we shall see how Tait used this approach to obtain what he called "physical proofs" of analytic statements.

Tait moved to Edinburgh in 1860, where he began a collaboration with William Thomson, then at Glasgow. In 1866 and 1867 their collaboration was at its peak, as they prepared their *Treatise on Natural Philosophy* (which was to become the standard introductory physics text in Britain for decades). Early in 1867, Tait showed Thomson an experimental demonstration of the stability properties of vortices by means of smoke-rings, as well as Helmholtz's mathematical treatment of the problem. Thomson described this event to Helmholtz in a letter:

> Just now, however, vortex motions have displaced everything else, since a few days ago Tait showed me in Edinburgh a magnificent way of producing them. Take one side (or a lid) off a box (any old packing box will serve) and cut a large hole in the opposite side. Stop the open side *AB* loosely with a piece of cloth, and strike the middle of the cloth with your hand. If you leave anything smoking in the box, you will see a magnificent ring shot out by every blow.

Thomson then went on to describe what he found particularly interesting about the phenomenon and the theory.

> The absolute permanence of the rotation, and the unchangeable relation you have proved between it and the portion of the fluid once acquiring such motion in a perfect fluid, shows that if there is a perfect fluid all through space, constituting the substance of all matter, a vortex-ring would be as permanent as the solid hard atoms assumed by Lucretius and his followers (and predecessors) to account for the permanent properties of bodies... thus if two vortex rings were once created in a perfect fluid, passing through one

another like links of a chain, they could never come into collision, or break one another, they would form an indestructible atom. [13]

Thomson embarked on the mathematical theory of these apparently indestructible vortex atoms at once, and his results were read before the Royal Society of Edinburgh a little over three weeks later. His paper *On Vortex Motion*, much augmented, appeared in 1878 [14]. Here he encountered and solved the problem posed by Helmholtz of extending Green's theorem to multiply-connected regions. For in order to investigate the properties of vortex atoms, it was necessary to solve boundary value problems where complicated vortices (such as those pictured) formed part of the boundary. (See FIGURE 2.)

In this paper, Thomson wrote the original version of Green's theorem thus:

$$\begin{aligned}\int_R \nabla\varphi\cdot\nabla\varphi'\,dV &= \oint \varphi\frac{\partial\varphi'}{\partial n}\,da - \int_R \varphi\nabla^2\varphi'\,dV\\ &= \oint\varphi'\frac{\partial\varphi}{\partial n}\,da - \int_R\varphi'\nabla^2\varphi\,dV.\end{aligned}$$

Here φ and φ' must be single-valued. Thomson then investigated what happens if φ' is multivalued, that is to say, if we consider R to be a multiply-connected region, and then considered how to set the problem up in general. A multiply-connected space can be made simply connected by making cuts, or by inserting what Thomson calls stopping barriers. The integral around an (almost) closed path from one side of the barrier to the other has a constant value k_i, which is the same for all such paths; such constants k_i exist for all stopping barriers and the integral in question becomes

$$\begin{aligned}\int_R \nabla\varphi\cdot\nabla\varphi'\,dV &= \oint\varphi\frac{\partial\varphi'}{\partial n}\,d\sigma + \sum_i k_i\iint\frac{\partial\varphi'}{\partial n}\,d\sigma' - \int_R\varphi\nabla^2\varphi'\,dV\\ &= \oint\varphi'\frac{\partial\varphi}{\partial n}\,d\sigma + \sum_i k_i\iint\frac{\partial\varphi}{\partial n}\,d\sigma' - \int_R\varphi'\nabla^2\varphi\,dV.\end{aligned}$$

Here $d\sigma'$ represents a surface element of the barrier surface.

Thus armed, Thomson was able to examine fluid motion in multiply-connected regions, concluding that the normal component of velocity of a fluid at every point of

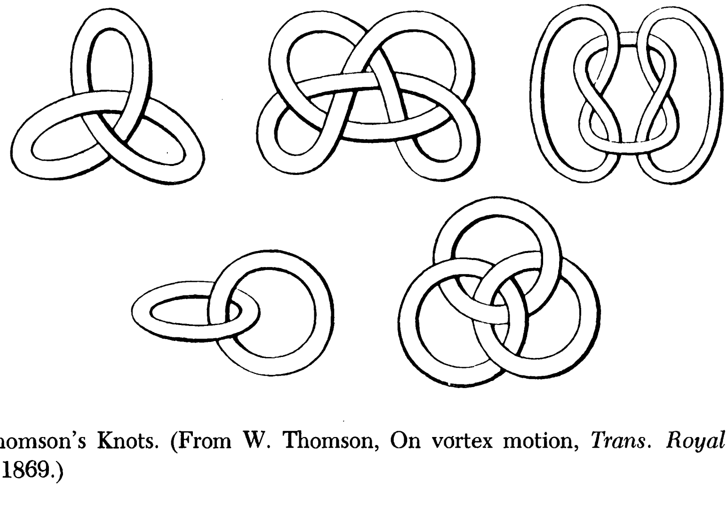


FIGURE 2.
William Thomson's Knots. (From W. Thomson, On vortex motion, *Trans. Royal Soc. Edinburgh*, 25, 1869.)

the boundary determines the motion inside a multiply-connected region (provided we know the circulation of the fluid in each region). He then considered how best to define the order of connectivity, noting that for some of the surfaces shown the stopping barriers must be self-intersecting and difficult to distinguish. He therefore proposed a definition using what he called "irreconcilable paths"—which we would now call homotopy classes of closed paths with base point. He selected a point on the surface, and noted that the connectivity of the surface is determined if we see how many mutually irreconcilable paths can be drawn on its surface. For a simply-connected region, for example, all closed paths on the surface are homotopic. Although this affords an unambiguous definition of connectivity, it is not easily possible using this method to get the generalization of Green's theorem. For that the stopping barriers are required.

Thomson's interest in vortex atoms thus led him directly to a generalization of Green's theorem, and to the question of the proper definition of connectivity. His proof technique is in essence the same as that of Green, augmented by the stopping barriers, and it is this method that is usually taught today in courses on vector calculus.

Tait's Quaternion Version of Green's Theorem

By the late 1860's, Tait's interest in quaternions had turned into a crusade. To the British association in 1871, he said:

> comparing a Cartesian investigation... with the equivalent quaternion one... one can hardly help making the remark that they contrast even more strongly than the decimal notation with the binary scale or with the old Greek arithmetic, or than the well-ordered subdivision of the metrical system with the preposterous non-systems of Great Britain.

In the same address, Tait pointed out that from the quaternion point of view:

> Green's celebrated theorem is at once seen to be merely the well-known equation of continuity expressed for a heterogeneous fluid, whose density at every point is proportional to one electric potential, and its displacement or velocity proportional to and in the direction of the electric force due to another potential. [**15**]

Let us see exactly what he means. In his 1870 paper *On Green's and Other Allied Theorems*, Tait supposed a spatial region R to be uniformly filled with points [**16**]. If points inside and outside the regions are displaced by a vector then we may have a net decrease or increase of the volume—that is, of the number of points—in the region R. This can be calculated in two ways:

1. We can find the total increase in density throughout R

$$\int_R \operatorname{div} \sigma \, dV \left(\text{in Tait's notation } \iint \mathrm{S} \cdot \nabla \sigma \, ds \right). \tag{6}$$

2. We can estimate the excess of those that pass inwards through the surface over those that pass outwards:

$$\iint \sigma \cdot \vec{n} \, da \left(\text{in Tait's notation } \iint s \cdot \sigma U V ds \right). \tag{7}$$

The expressions (6) and (7) must be equal, yielding what we now call the divergence theorem from the equation of continuity. If we consider that the density—for example, of electric fluid— is given by a potential P_1, and the displacement is proportional to a force σ with potential P, then we have:

$$\nabla(PP_1) = P\nabla P_1 + P_1\nabla P$$

and

$$\nabla^2(PP_1) = P\nabla^2 P_1 + P_1\nabla^2 P + 2(\nabla P \cdot \nabla P_1). \tag{8}$$

But by the divergence theorem

$$\int_R \nabla^2(PP_1)\, dV = \int_R \operatorname{div}(\nabla PP_1)\, dV = \int_{\partial R} (\nabla PP_1) \cdot \vec{n}\, da$$

$$= \int_{\partial R} (P\nabla P_1 + P_1\nabla P) \cdot \vec{n}\, da.$$

Hence from (8)

$$\int_{\partial R} (P\nabla P_1 + P_1\nabla P) \cdot \vec{n}\, da = \int_R \left(P\nabla^2 P_1 \nabla^2 P\right) dV + 2\int_R \nabla P \cdot \nabla P_1\, dV.$$

But the left side here, by the divergence theorem, is

$$= \int_R \left(P\nabla^2 P_1 - P_1\nabla^2 P\right) dV.$$

Combining these two yields Green's theorem in the form

$$\int_R (\nabla P \cdot \nabla P_1)\, dV = -\int_R P_1\nabla^2 P + \int_{\partial R} P\frac{\partial P}{\partial n}\, da$$

$$= -\int_R P\nabla^2 P_1 + \int_{\partial R} P\frac{\partial P_1}{\partial n}\, da.$$

Notice that the argument depends on treating geometric points as mobile physical entities, with continuity properties like those of a fluid.

We find Tait's views nicely summarized in his 1892 review of Poincaré's *Thermodynamique*:

> Some forty years ago, in a certain mathematical circle at Cambridge, men were wont to deplore the necessity of introducing words at all in a physico-mathematical textbook: the unattainable, though closely approachable Ideal being regarded as a world devoid of aught but formulae! But one learns something in forty years, and accordingly the surviving members of that circle now take a very different view of the matter. They have been taught alike by experience and by example to regard mathematics, so far at least as physical enquiries are concerned, as a mere auxiliary to thought... this is one of the great truths which were enforced by Faraday's splendid career. [17]

Conclusion

Our excursion from Green to Tait has taken us from electrostatics and potential theory, via complex analysis and fluid dynamics, to homotopy classes of maps and vector analysis. While I have only touched on a few of the interesting problems associated with these developments, I hope that I have shown that physical thinking is important, not only in posing mathematical problems, but also in solving them. In particular, physical thinking may lead to the creation of certain mathematical notions, such as connectivity, which are of interest in their own right, for example in the classification of the knots described by Thomson. Tait undertook this classification problem around 1870, achieving the first basic results of knot theory.

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