degrees. The maximal subgraphs with maximum degree 3 are much harder to characterize than those with maximum degree 2 which are unions of cycles together with at most one isolated point or with at most one pair of connected points.

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Pólya's Geometric Picture of Complex Contour Integrals

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Complex contour integrals contain an element of mystery which has troubled me since my student days. Churchill and Brown [**1**] express the problem succinctly: "Definite integrals in calculus can be interpreted as areas, and they have other interpretations as well. Except in special cases, no corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane." In 1974, George Pólya suggested a simple solution, but his idea does not seem to be widely appreciated. Computer graphic techniques can be used to help students visualize and estimate complex integrals once Pólya's approach is adopted.

In classical potential theory it has long been the custom to associate to a real harmonic function $u(x, y)$ a "complex potential" $f(x+iy) = u(x, y) + iv(x, y)$, where $v(x, y)$ is a harmonic conjugate of $u(x, y)$. Then the real and imaginary parts of $\overline{f'(z)}$ are the components of the gradient field $\langle u_x(x, y), u_y(x, y)\rangle$ corresponding to the potential $u(x, y)$. Pólya's idea was simply this: to any complex function $f(x+iy) = u(x, y) + iv(x, y)$, associate the plane vector field $\overline{f(x+iy)} = \langle u(x, y), -v(x, y)\rangle$, rather than the derived field $\overline{f'(x+iy)}$. In [**2**] it is shown that complex integrals with integrand $f(z)$ have a simple geometric and physical interpretation in terms of the associated vector field $\overline{f(x+iy)}$. Our goal here is to spread this gospel, showing that the vector field picture can be used to estimate specific contour integrals, and leading to new insight into the theory of complex integration. An earlier paper [**3**] indicates the usefulness of the vector field picture of complex functions (as an alternative to the traditional view of a function as a mapping on the complex plane), in analyzing zeros and singular points of complex functions.

To emphasize the distinction between a complex function and its associated vector field, we henceforth write $\overline{\mathbf{W}}(z)$ or $\overline{\mathbf{W}}(x, y)$ to denote the Pólya vector field corresponding to a complex function $f(z)$. Thus if $f(x+iy) = u(x, y) + iv(x, y)$ is the decomposition of $f(z)$ into its real and imaginary parts, then $\overline{\mathbf{W}}(x, y) = \langle w_1(x, y), w_2(x, y)\rangle$ with $w_1 = u$, $w_2 = -v$.

The integral of f over an oriented curve γ can be expressed in terms of real integrals of the components of $\overline{\mathbf{W}}$ along γ:

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+i\,dy) = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy$$

$$= \int_\gamma w_1\,dx + w_2\,dy + i\int_\gamma w_1\,dy - w_2\,dx = \int_\gamma \overline{\mathbf{W}}\cdot\mathbf{T}\,ds + i\int_\gamma \overline{\mathbf{W}}\cdot\mathbf{N}\,ds,$$

where $\mathbf{N}$ is the normal vector obtained by turning the unit tangent vector $\mathbf{T}$ *clockwise* through $\pi/2$. In words, the real part of $\int_\gamma f(z)\,dz$ is the integral of the tangential component of the Pólya vector field $\overline{\mathbf{W}}$ over γ (the flow along γ, if we picture $\overline{\mathbf{W}}$ as a velocity field); and the imaginary part of $\int_\gamma f(z)\,dz$ is the integral of the normal component of $\overline{\mathbf{W}}$ over γ (the *flux* across γ). An immediate payoff of this geometric interpretation is that it makes clear the fact that the value of a contour integral is independent of the parametrization, and changes sign if the orientation of the curve is reversed.

Just as one can estimate a real integral $\int_a^b f(x)\,dx$ by interpreting it as the signed area between the graph of f and the interval $[a, b]$ on the x-axis, a complex integral $\int_\gamma f(z)\,dz$ can be roughly approximated by visually estimating the flow and flux of the Pólya vector field $\overline{\mathbf{W}}$ along the path.

In FIGURE 1, for example, the vector field $\overline{\mathbf{W}}$ for the function $f(z) = 1/z$ is shown along the unit circle. The vector $\overline{\mathbf{W}}(z)$ is normal to the path at each point z, so the flow of $\overline{\mathbf{W}}$ along the contour is zero. The normal component of $\overline{\mathbf{W}}$ is apparently constant, namely 1, so the flux of $\overline{\mathbf{W}}$ across the path is simply this constant times the length of the path, viz., 2π. Thus our geometric analysis has shown that $\int_{|z|=1} \frac{1}{z}\,dz = 2\pi i$.

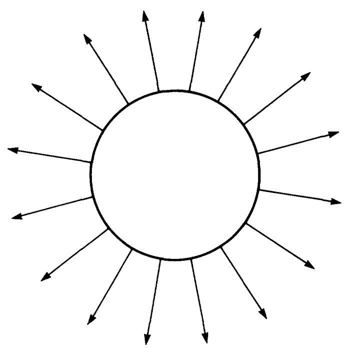

FIGURE 1. **Pólya vector field for $f(z) = 1/z$ on the unit circle.**

The example just considered is of course very special; in general one can only estimate the integrals of the tangential and normal components of $\overline{\mathbf{W}}$ over γ, from a plot of the vector field along this path. To emphasize how closely the procedure for estimating complex integrals parallels that for estimating real integrals, we ask the reader's indulgence as we briefly recall the latter.

To estimate $\int_a^b f(x)\,dx$ from a sketch of the graph of f over $[a, b]$, of course, one estimates the area between the graph and the x-axis, and subtracts the area below the axis from the area above. In more detail, we might mentally form a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that $f(x)$ does not change sign on each subinterval; then estimate each of the integrals $\int_{x_k}^{x_{k+1}} f(x)\,dx$, and add the resulting signed numbers. To estimate the area between the graph and the x-axis over each subinterval $[x_k, x_{k+1}]$, we estimate the mean height $\overline{y_k}$ of the graph over this subinterval and use the product $\overline{y_k}(x_{k+1} - x_k)$ as our estimate of the area.

For example the mental process used in estimating $\int_0^5 f(x)\,dx$ for the function graphed in FIGURE 2 might go something like this: consider the partition $0 < 2 < 3 < 5$;

$$\int_0^2 f(x)\,dx \cong (1.1)(2-0), \qquad \int_2^3 f(x)\,dx \cong (-.1)(3-2), \qquad \int_3^5 f(x)\,dx \cong (.7)(5-3),$$

so $\int_0^5 f(x)\,dx \cong 2.2 - .1 + 1.4 = 3.5$. If an analytical evaluation produced a far different value for this integral, say -3, we would know an error had been made; the simplicity of the geometric estimate makes it very convincing.

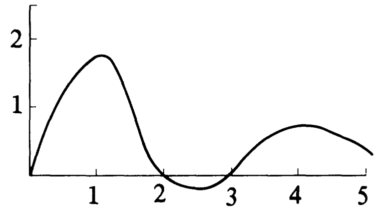

FIGURE 2

The vector field interpretation of complex integrals can be used in a similar way to provide a simple estimate based on geometric intuition, which can then be used as a check against analytical methods.

To estimate $\int_\gamma f(z)\,dz$, we must separately estimate its real part $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{T}\,ds$ and its imaginary part $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{N}\,ds$. To estimate $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{T}\,ds$, we first partition the curve into segments $\gamma_1, \gamma_2, \dots, \gamma_n$ on which $\overline{\mathbf{W}} \cdot \mathbf{T}$ has constant sign (recall that $\overline{\mathbf{W}} \cdot \mathbf{T}$ is positive just if the angle between $\overline{\mathbf{W}}$ and $\mathbf{T}$ is acute). Then on each segment γ_k we visually estimate the mean tangential component τ_k of $\overline{\mathbf{W}}$, such that $\tau_k l_k \cong \int_{\gamma_k} \overline{\mathbf{W}} \cdot \mathbf{T}\,ds$, where l_k denotes the length of γ. In practice the plot of $\overline{\mathbf{W}}$ along γ is scaled, i.e., there is a factor SCALE such that a vector of apparent length 1 in the plot represents a vector in $\mathbb{C}$ with the same direction but of magnitude SCALE. Thus if τ_k denotes the *apparent* mean tangential component of $\overline{\mathbf{W}}$ along the curve segment γ_k, then $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{T}\,ds \cong \text{SCALE} \sum_{k=1}^{n} \tau_k l_k$. The procedure for estimating $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{N}\,ds$ is similar.

EXAMPLE 1. If the plot of $\overline{\mathbf{W}}$ along γ were as indicated in FIGURE 3, to estimate $\int_\gamma f(z)\,dz$ we might reason as follows.

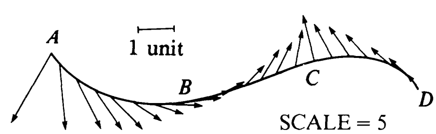

FIGURE 3

On segment AC the angle between $\overline{\mathbf{W}}$ and $\mathbf{T}$ is acute, so the tangential component of $\overline{\mathbf{W}}$ is positive on this segment. At A the (apparent) length of $\overline{\mathbf{W}}$ is about 2 units, and the tangential component (projection of $\overline{\mathbf{W}}$ onto the tangent line) is about 1.5 units. The vectors $\overline{\mathbf{W}}$ decrease in length as we move toward B, but they become more nearly parallel to $\mathbf{T}$, so the tangential component of $\overline{\mathbf{W}}$ decreases only to about .5 units. If we estimate the mean tangential component of $\overline{\mathbf{W}}$ to be 1 unit along this segment, then since the length of the arc AB is about 5 units, we estimate $\tau_1 l_1 \cong (1)(5) = 5$. The vectors $\overline{\mathbf{W}}$ increase in length from B to C, but the tangential component decreases from about .5 at B to 0 at C. Using an estimate $\tau_2 \cong .3$ for the mean tangential component on BC, and estimating the length of this segment to be 3 units, gives $\tau_2 l_2 \cong (.3)(3) = .9$. On segment CD the tangential component of $\overline{\mathbf{W}}$ starts at 0, becomes negative with a minimum of about $-.2$, and finally returns to 0 at D. We estimate $\tau_3 l_3 \cong (-.1)(3) = -.3$, and since the scale factor for the plot is SCALE = 5, our estimate of $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{T}\,ds$ would be

$5(5 + .9 - .3) = 28$. Similarly estimating the mean normal components of $\overline{\mathbf{W}}$ along AB, BC, and CD to be $\nu_1 \cong .5$, $\nu_2 \cong -.4$, and $\nu_3 \cong -.5$ gives SCALE $\sum \nu_k l_k \cong 5[(.5)(5) + (-.4)(3) + (-.5)(3)] = -1$. So if $\overline{\mathbf{W}}$ were the Pólya vector field of a complex function $f(z)$, our geometric estimates indicate that $\int_\gamma f(z)\,dz \cong 28 - i$. We cannot be certain of the sign of the imaginary part $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{N}\,ds$, since small errors in estimating the ν_k and l_k could affect the sign of the sum, but we can say with assurance that the flow along γ is positive (about 30), and the flux across γ is near 0. If an analytical calculation led to the result $\int_\gamma f(z)\,dz = 2\pi i$, we would have good reason to recheck the analysis.

In an introductory course on complex analysis the main "application" of complex integration is to evaluate certain real integrals using residue calculus. Typically one completes the real interval of integration to a closed contour in the complex plane, applies the residue theorem to evaluate an appropriate complex integral over this contour, and then tries to determine the contribution to the total produced by the integral along the real axis. By looking at a plot of the Pólya vector field along the contour, this last step can sometimes be clarified. (The residue at a simple pole also can be estimated geometrically, but showing how this may be done would take us too far from our main theme here.)

EXAMPLE 2. Evaluate

$$I_1 = \int_0^\infty \frac{1}{x^3+1}\,dx.$$

One first uses the residue theorem to evaluate

$$I = \int_\gamma \frac{1}{z^3+1}\,dz, \qquad \gamma = \gamma_1 + \gamma_R + \gamma_2,$$

where γ_1 follows the real axis from the origin to R, γ_R is the arc of the circle $|z| = R$ from R to $Re^{2\pi i/3}$, and γ_2 is the line segment from $Re^{2\pi i/3}$ back to the origin. $I = 2\pi i\,\mathrm{Res}(f, z_1)$, where $f(z) = 1/(z^3+1)$ and z_1 is the simple pole of f at $e^{\pi i/3}$. We calculate

$$\mathrm{Res}(f, z_1) = \lim_{z \to z_1} (z - z_1) f(z) = \frac{1}{\left[\frac{3}{2} + i\frac{\sqrt{3}}{2}\right](i\sqrt{3})}, \quad \text{so } I = \pi\left[\frac{1}{\sqrt{3}} - i\left(\frac{1}{3}\right)\right].$$

Because $|f(z)|$ decreases more rapidly than $\dfrac{1}{|z^2|}$ as $|z|$ increases, $\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = 0$. So since the value of I is independent of R, $I = I_1 + I_2$, where $I_2 = \lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz$.

Now the values of z^3 at $z = t$ on γ_1 and $z = te^{2\pi i/3}$ on γ_2 are identical, so the Pólya vectors $\overline{\mathbf{W}}(t)$ and $\overline{\mathbf{W}}(te^{2\pi i/3})$ are equal. However, $\overline{\mathbf{W}}(t)$ is directed along the path γ_1, whereas $\overline{\mathbf{W}}(te^{2\pi i/3})$ makes an angle of $\pi/3$ with the unit tangent vector $\mathbf{T}$ to γ_2. [See FIGURE 4.] So the tangential component of $\overline{\mathbf{W}}(te^{2\pi i/3})$ is

$$|\overline{\mathbf{W}}(t)|\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}|\overline{\mathbf{W}}(t)|,$$

and the normal component is

$$|\overline{\mathbf{W}}(t)|\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}|\overline{\mathbf{W}}(t)|.$$

Thus

$$I_2 = \frac{1}{2}I_1 - i\frac{\sqrt{3}}{2}I_1, \quad \text{where } I_1 = \int_0^\infty \frac{1}{t^3+1}\,dt. \tag{$*$}$$

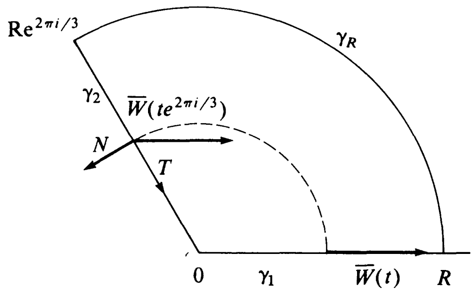

FIGURE 4

So

$$I = I_1 + I_2 = \left(\frac{3}{2} - i\frac{\sqrt{3}}{2}\right) I_1,$$

and comparing this with the value for I found above using the residue theorem, we conclude that $I_1 = \frac{2\pi}{3\sqrt{3}}$. Our discussion differs from the usual analytic evaluation of I in only one essential: rather than parametrizing γ_1 and deriving equation (*) by analytic means, our geometric argument uses the decomposition of $\overline{\mathbf{W}}(te^{2\pi i/3})$ into its tangential and normal components to derive this relationship between I_2 and I_1.

Besides clarifying the analysis of specific integrals, and throwing new light on familiar properties of complex functions, the vector field approach to complex integrals can lead to new theoretical results. The inequality $\left|\int_\gamma f(z)\,dz\right| \leqslant \int_\gamma |f(z)|\,ds$ is fundamental in estimating complex integrals, but does not seem to have a generally accepted name. It is sometimes called the *triangle inequality* for complex integrals, because it may be viewed as a consequence of the fact that the straight line segment is the shortest distance between two points in the complex plane. I have been unable to find any discussion in the literature of the conditions under which equality holds in this triangle inequality. The reason seems to be that the appropriate condition is not conveniently expressible in terms of the mapping properties of complex functions. But from the vector field point of view the condition is beautifully simple. Note that, because the value of a contour integral is unaffected when the path of integration is changed by a continuous deformation (keeping the endpoints fixed) in the region of analyticity of the integrand, the conditions for equality in the triangle inequality will involve both the contour γ and the integrand $f(z)$.

THEOREM. *Let $f(z)$ be a continuous complex function on a domain containing the piecewise differentiable arc γ. Then equality holds in the triangle inequality*:

$$\left|\int_\gamma f(z)\,dz\right| = \int_\gamma |f(z)|\,ds$$

exactly when the Pólya vector field $\overline{\mathbf{W}}$ makes a constant angle with the tangent vector field $\mathbf{T}$ *along* γ.

Our proof is based on a simple lemma about the modulus of a vector sum, and its continuous analogue for vector integrals.

LEMMA 1. *If* $\mathbf{W} = \sum_{k=1}^{n} \mathbf{V}_k$, *then* $|\mathbf{W}| = \sum_{k=1}^{n} |\mathbf{V}_k| \cos\theta_k$, *where θ_k is the angle between* $\mathbf{V}_k$ *and* $\mathbf{W}$. [*In words, the sum of the components of the summands along the sum* $\mathbf{W}$ *gives the modulus of the sum*, $|\mathbf{W}|$.]

Proof.

$$\sum_{k=1}^{n} |\mathbf{V}_k| \cos\theta_k = \sum_{k=1}^{n} \frac{\mathbf{V}_k \cdot \mathbf{W}}{|\mathbf{W}|} = \frac{1}{|\mathbf{W}|} \sum_{k=1}^{n} \mathbf{V}_k \cdot \mathbf{W} = \frac{1}{|\mathbf{W}|} \mathbf{W} \cdot \mathbf{W} = |\mathbf{W}|.$$

LEMMA 2. If $\mathbf{V}(t)$, $a \leqslant t \leqslant b$, is any continuous vector function, and $\mathbf{W} = \int_a^b \mathbf{V}(t)\,dt$, then $|\mathbf{W}| = \int_a^b |\mathbf{V}(t)| \cos\theta(t)\,dt$, where $\theta(t)$ is the angle between $\mathbf{V}(t)$ and $\mathbf{W}$.

Proof. Let $R_n = \sum_{k=1}^{n} \mathbf{V}(t_k)\Delta t$ denote a Riemann sum approximation to $\mathbf{W}$ relative to the partition of $[a,b]$ into n equal subintervals of length $\Delta t = (b-a)/n$. By Lemma 1, $|R_n| = \sum_{k=1}^{n} |\mathbf{V}(t_k)| \cos\theta_k \Delta t$, where θ_k is the angle between $\mathbf{V}(t_k)$ and R_n. As $n \to \infty$, $R_n \to \mathbf{W}$, so given any $\varepsilon > 0$, by taking n sufficiently large we can make $|\cos\theta_k - \cos\theta(t_k)| < \dfrac{\varepsilon}{M(b-a)}$ for all k, where $\theta(t_k)$ is the angle between $\mathbf{V}(t_k)$ and $\mathbf{W}$, and $M = \max_{t\in[a,b]} |\mathbf{V}(t)|$. Then

$$\left|\sum_{k=1}^{n} |\mathbf{V}(t_k)|(\cos\theta_k - \cos\theta(t_k))\Delta t\right| \leqslant \sum_{k=1}^{n} |\mathbf{V}(t_k)| \frac{\varepsilon}{M(b-a)}\Delta t \leqslant nM\frac{\varepsilon}{M(b-a)} \cdot \frac{b-a}{n} = \varepsilon.$$

That is, $\sum |\mathbf{V}(t_k)| \cos\theta_k \Delta t \to \sum |\mathbf{V}(t_k)| \cos\theta(t_k)\Delta t$ as $n \to \infty$. But the left side approaches $|\mathbf{W}|$, since $R_n \to \mathbf{W}$, and the right is a Riemann sum approximation to $\int_a^b |\mathbf{V}(t)| \cos\theta(t)\,dt$. So $\mathbf{W} = \int_a^b |\mathbf{V}(t)| \cos\theta(t)\,dt$, as claimed.

COROLLARY. *If* $\mathbf{V}(t)$ *is continuous on* $[a,b]$, *then* $\left|\int_a^b \mathbf{V}(t)\,dt\right| \leqslant \int_a^b |\mathbf{V}(t)|\,dt$, *with equality exactly when* $\mathbf{V}(t)$ *has constant polar angle.*

Proof. If $\theta(t)$ is the angle between $\mathbf{V}(t)$ and $\mathbf{W} = \int_a^b \mathbf{V}(t)\,dt$, then since $1 - \cos\theta(t) \geqslant 0$ throughout $[a,b]$, we have $0 \leqslant \int_a^b |\mathbf{V}(t)|\{1 - \cos\theta(t)\}\,dt$, with equality exactly when $\cos\theta(t) \equiv 1$. So, using Lemma 2, $\left|\int_a^b \mathbf{V}(t)\,dt\right| = |\mathbf{W}| = \int_a^b |\mathbf{V}(t)| \cos\theta(t)\,dt \leqslant \int_a^b |\mathbf{V}(t)|\,dt$, with equality just if the angle $\theta(t)$ between $\mathbf{V}(t)$ and $\mathbf{W}$ is zero, which is equivalent to the requirement that $\mathbf{V}(t)$ have constant polar angle.

Proof of the theorem.

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right| \leqslant \int_a^b |f(z(t))||z'(t)|\,dt = \int_a^b |f(z)|\,ds,$$

with equality just if the vector function $f(z(t))z'(t)$ has constant polar angle. But the polar angle of $f(z(t))z'(t)$ is $\arg f(z(t)) + \arg z'(t)$, or $\arg z'(t) - \arg\overline{f(z(t))}$, which we recognize as the angle between the tangent vector to the path and the Pólya vector $\overline{\mathbf{W}}$ at $z(t)$.

Note that the radial vector field for $f(z) = 1/z$ in FIGURE 1 makes a constant angle $\pi/2$ with the tangent vector field on the circle $|z| = 1$, as required in the theorem. And indeed

$$\left|\int_{|z|=1} \frac{1}{z}\,dz\right| = \int_{|z|=1} \left|\frac{1}{z}\right|\,ds,$$

the common value being 2π. Another example where the constant-angle hypothesis is satisfied is the integral $\int_{\gamma_2} 1/(z^3+1)\,dz$ discussed in Example 2; and the fact that equality holds in the triangle inequality for this integral is immediate from equation $(*)$ of that Example.

In view of the availability of microcomputers* with powerful graphics capabilities, and the increasing number of students familiar with such hardware, it becomes feasible to make class assignments involving use of Pólya vector field pictures in an introductory complex analysis course. Experience with such a geometric model, especially in the study of contour integration, can help eliminate from complex analysis the undesirable connotations of the term "imaginary."

References

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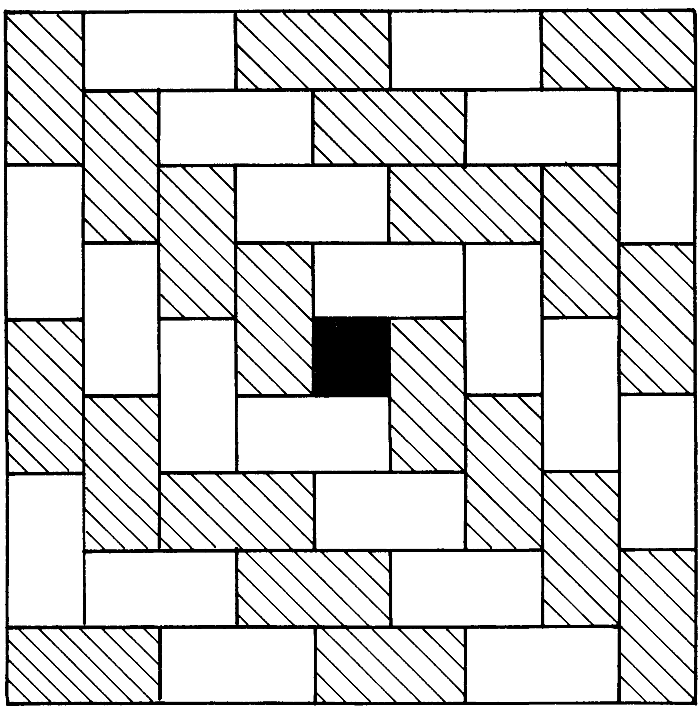
*An advantage of mainframe computers in this connection is their access to mathematical libraries for evaluating functions of a complex variable. A simple program (in FORTRAN 77, for a CalComp plotter) to sketch the Pólya vector field for an arbitrary function $f(z)$ along any specified contour, is available from the author upon request.

Proof without Words:
1 Domino = 2 Squares: Concentric Squares



$$1 + 4 \cdot 2 + 8 \cdot 2 + 12 \cdot 2 + 16 \cdot 2 = 9^2$$

$$1 + 2 \sum_{k=1}^{n} 4k = (2n+1)^2.$$

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