ARTICLES

Parliamentary Coalitions: A Tour of Models

Mathematics explores the relationship between politics and rationality.

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The study of government and politics, like all the other social sciences, has made increasing use of mathematics in the last thirty years. This has involved not only the use of statistics and computers to increase the scope and thoroughness of empirical work, but also the development of mathematical models in political theory. Through the use of models, mathematics has come to play an important role in political science at the conceptual level. This development has not been greeted with universal enthusiasm by practitioners in the field. An academic colleague of one of the authors, in a department of Government, has been heard to complain with some bitterness that he can no longer read the *American Political Science Review* because it is full of obscure mathematical symbols. Non-enthusiasts have their point. Politics often seems far removed from the domain of precise, rational thought which mathematics epitomizes. Nevertheless, mathematics *has* been able to contribute insights to the study of political systems. In this article we explore the use of mathematical models in various attempts to understand one of the central concerns of politics—the ways in which politicians go about forming coalitions with other politicians.

In the first section we introduce the problem of coalition formation in the context of parties trying to form a coalition government in a parliamentary system. This will be our central focus in the article. We start by considering some of the earliest and simplest models, which lead to a geometric formulation of the problem. In the second section we discuss two more complex models from the tradition of mathematical game theory. We have tried to illustrate the methods of thinking involved in these models without presenting all of the formal details (for which we give references). In the next section we return to a simpler "dynamic" model of coalition formation and explore its properties. Finally, we have some general words to say about the problems of testing models in the social sciences, and a "coda" in which we invite you to think about how a coalition model might be useful outside of political science.

The problem of coalition formation and early models

The motivating problem for much of the mathematical work on coalition theory, and the data base for most of the empirical work until quite recently, is the problem of understanding the formation of governing coalitions in parliamentary democracies. Most of the data is from western Europe. Let's look at a particular situation, in order to have a specific example before us.

After the 1965 parliamentary elections in Norway, the five major parties had won the following numbers of seats:

Party	Number of seats
A. Labor	68
B. Christian	13
C. Liberal	18
D. Center	18
E. Conservative	31
	148

Since no party had the necessary majority of 75, a majority government could be formed only by a coalition of two or more parties. Which coalition would we expect to form? The goal in positive political theory (see, for example, [**23**], [**24**], or [**1**]) is to build a model to explain which coalition should form in this and similar situations, on the basis of some kind of rational choices by the actors (here parties) involved. We would then test the model against the large amount of data on twentieth century parliamentary coalitions. If the fit is good, we think we have a deeper and more general understanding of parliamentary coalition formation than would be obtained by, say, case-by-case study or exhaustive descriptive classification.

To build a rational choice model, we need to posit some kind of goals for the actors involved. We think they might then act to try to reach those goals. The earliest models posited the following goal:

Goal P. *There is a positive payoff associated to the act of governing. This payoff will be shared in some way among the members of a governing coalition. Parties wish to maximize their share of this payoff.*

Notice that we need not be specific about the nature of the payoff. It might be something like "power," that nebulous entity which *homo politicus* might value for its own sake, as opposed to

"There is a positive payoff associated to the act of governing."

valuing it for the sake of attaining other goals. It might be some kind of "spoils" of governing, say in the form of patronage appointments. However, it is something divisible which can be shared. It isn't related to political ideology or, at least directly, to the ability to implement certain governing policies. This goal has the advantage of being minimalist. The idea is to see how well we can explain coalition formation using only this goal. If we fail, we will want to consider other possible goals.

The most direct consequence of this model was most clearly enunciated by William Riker [**23**] in the late 1950s. If the payoff is to be shared among the members of a governing coalition with each member wanting as much as possible, then the coalition which forms should have no superfluous members—no members whose deletion would leave the coalition still of winning size. If there were such a member, all of the other members would be better off expelling him and dividing his share of the payoff among themselves. In other words, we have the

Riker Minimal Winning Coalition Principle. *Only a minimal winning coalition should form.*

In the Norwegian example there are sixteen winning coalitions, but only five minimal winning coalitions: AB, AC, AD, AE, and $BCDE$. The minimal winning coalition principle says that only one of these five should form. It thus makes a testable prediction, but the prediction is fairly weak, since it doesn't say which one of the five should form. If we make stronger assumptions about how the sharing of the spoils might be done, we can make stronger predictions.

In the late 1950s sociologists experimenting with three-person coalition formation found that people often acted as if they believed that payoffs might be divided in proportion to the resources which each person brought to the coalition [**6**], [**12**]. In our situation, the resources are parliamentary votes. We thus might posit that payoffs will be divided in proportion to the number of parliamentary seats each party has. Riker used this idea to make a more specific prediction:

Riker Least Resources Principle. *The coalition which forms should be the winning coalition with the smallest total number of votes.*

The idea here is that all members of this coalition will prefer it to any other winning coalition because it will give them a larger share of the spoils. For example, consider our Norwegian parties.

Minimal winning coalition	Number of votes
AB	81
AC	86
AD	86
AE	99
$BCDE$	80

The least resources principle predicts that coalition $BCDE$ will form. Party B will prefer this coalition, which will give it $13/80$ of the spoils, to coalition AB, which will give it only $13/81$. Similarly, C, D and E will all prefer this coalition to any other winning coalition. Party A prefers AB ($68/81$ of the spoils), but can't implement this coalition because B prefers $BCDE$.

Of course, we could make other assumptions about how spoils would be divided. For instance, if we assumed they would be divided *equally*, on the grounds that all members of any minimal winning coalition are equally important (since the defection of any one of them would cause the coalition to lose), we would get the

Fewest Actor Principle [**18**]. *The coalition which forms should have the fewest members of all winning coalitions.*

In the Norwegian example, this principle would predict AB, AC, AD or AE instead of $BCDE$. The fewest actor principle might also be attractive if there are large **transaction costs** in forming coalitions. It costs more time, negotiating energy, and uncertainty to put together a four-member coalition than to agree on a two-member coalition.

It's time to come down to earth and look at the facts. When we look at the data on coalition formation in parliamentary systems, we find that ([3], [10], [11], [19])

i) although there are a number of countries which are exceptions, in most countries minimal winning coalitions form much of the time;
ii) neither the least resources principle nor the fewest actor principle predicts with better than chance accuracy.

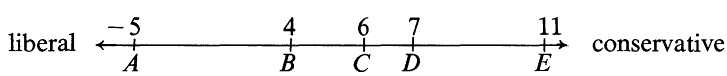
Parties often form minimal winning coalitions, but not necessarily least resource or fewest actor coalitions.

We might note that there are situations in which not even the minimal winning coalition theory is supported. On the one hand, it may not be necessary for a governing coalition to be winning: minority coalition governments have not been uncommon in western democracies [19]. On the other hand, winning coalitions on legislative bills are often larger than minimal winning. One reason seems to be *uncertainty*. In building a large, temporary coalition, you are never quite sure who is with you and who is not, so you need to allow a margin for error (see [23] or [24] for a discussion).

Apparently we need to bring in other goals if we want to build a model which will make more exact predictions than the minimal winning coalition principle. The most important factor which we have not yet considered, and one of the staples of political analysis, is the role of political ideology. Beyond the desire for power or patronage, it is generally believed that political parties do have policies which they would like to see implemented:

Goal I. *A party would like to see its policies implemented. Hence it wants to join a coalition with other parties whose values and ideological positions are close to its own.*

The first problem is how we are to model things like values and ideological positions. The most familiar model dates back to 1789, when parties in the French National Assembly with "radical," "liberal," "moderate," "conservative," and "reactionary" ideologies were seated from left to right in the chamber. In common political analysis it seems to be believed that we can often place parties on a one-dimensional left-right continuum based on their views about the extent to which government should intervene in the economy, redistributional issues, the desirability of legislating social behavior, and other issues. If we think this can be done, perhaps we can assign parties to points on the real line, representing their ideological position in this "ideological space." For instance, if we place the Norwegian parties by their stands on economic issues, we might get something like this [8]:



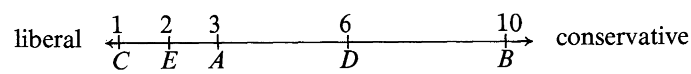
The weak (**ordinal**) assumption is that only the order of the parties along the line is meaningful. The stronger (**cardinal**) assumption is that relative distances between parties is meaningful: D is twice as close to C as B is, E is three times as close to D as A is, and so forth (see, for example, Chapter 8 of [25]).

In 1970 Robert Axelrod [3] noted that even with just the ordinal assumption, we can say that coalitions which are **connected** will have more commonality of values among their members than coalitions which are not connected. Here a coalition S is connected if whenever $X, Z \in S$ and $X < Y < Z$, then $Y \in S$. In the Norwegian example the connected winning coalitions are AB, ABC, $ABCD$, $BCDE$ and $ABCDE$.

Connected Coalition Principle. *The coalition which forms should be connected.*

We can sharpen this principle, as Axelrod did, by combining it with the minimal winning coalition principle to predict that a *connected minimal winning* coalition should form: AB or $BCDE$ in the Norwegian example.

The connected coalition principle and its minimal winning strengthening are both supported fairly well by parliamentary data [**3**], [**10**], [**19**]. For example, Abram DeSwann found that in a sample of 108 coalition governments, 55 were connected minimal winning. However, we can note two problems. The first is that there may be a number of connected minimal winning coalitions, and we might like to make a more precise prediction. The second is that sometimes it is quite difficult to place parties on a one-dimensional continuum. What do you do with a party which is "conservative" on government spending but "liberal" on social issues? As an example, cultural issues are important to many Norwegian voters, and if we locate parties by their stands on cultural issues we get something like [**8**]:



Now the connected minimal winning coalitions are AE and AD.

The most natural way out of this difficulty, for a mathematician at least, is to plot positions on both scales at once, i.e., to represent parties' ideological positions as points in a coordinate plane (see FIGURE 1). If there were three salient kinds of issues, we would plot parties as points in three-dimensional space. Notice that we now need a new definition of when a coalition is connected. The natural definition is that a coalition is connected if it includes any party within the **convex hull** of its members. (The convex hull of a set of points is the smallest convex set containing those points.) In FIGURE 1, AE or CDE or $BCDE$ are connected, but BCE is not connected since it doesn't include D, which is in its convex hull. For this definition to work, the relevant scales must be *cardinal* scales: if D were moved to $(8,6)$ the ordinal relations would be unchanged, but now BCE would be connected. In fact, it is traditional in this kind of "spatial modeling" in political science to make a much stronger assumption: that the relevant scales are **comparable** in the sense that *Euclidean distance* measures ideological proximity (see for example [**24**] or [**1**]). In FIGURE 1, $d(B,D)=5$ and $d(A,B)=11.4$, so A is more than twice as far from B as D is. This quite strong assumption could of course be relaxed, for instance, by developing the theories of the next sections in other metrics, but we will accept this assumption for our discussion.

Finally, notice that, with the expansion into two or more dimensions, the connected minimal winning coalition theory becomes distinctly less helpful. In the Norwegian example of FIGURE 1, *all* minimal winning coalitions are connected, and we would want to narrow down this class.

Party	Weight	Position
A. Labor	68	$(-5, 3)$
B. Christian	13	$(4,10)$
C. Liberal	18	$(6, 1)$
D. Center	18	$(7, 6)$
E. Conservative	31	$(11, 2)$
	148	

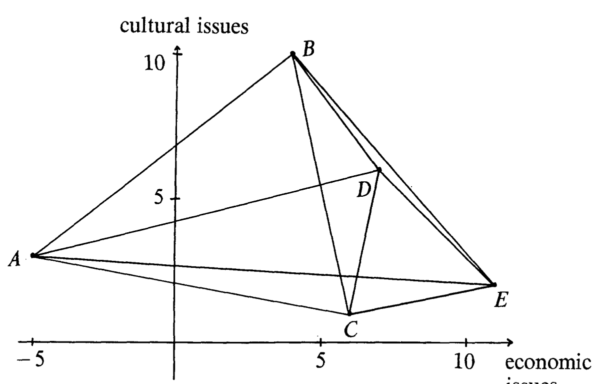


FIGURE 1. **Parties in Norway, 1965.**

Two game theoretic approaches to spatial coalitions

Our model is now as follows. We are given an n-dimensional Euclidean **ideological space**. Parties are represented as points in that space, and the distance between points represents the ideological proximity of the parties. Each party has a **weight** (its number of seats) and winning coalitions are those with large enough total weights (e.g., a majority of seats in the parliament). We can now be more specific about the goal that parties want to join a coalition which will pursue policies similar to theirs. We can think of any coalition, when it forms or thinks about forming, as choosing a point in the ideological space which represents the policies it would follow. Parties want to join coalitions which will choose points close to theirs. For example, suppose that in FIGURE 2 the five parties are approximately equally weighted, so that any three would be a winning coalition. If coalition GHJ formed and chose policy point α, its members would be fairly pleased, since α is fairly close to each of their policy points.

Now the question. In this situation, can we predict which coalition would form? This **spatial coalition theory** has been actively studied for the past ten years, and a number of different solutions have been proposed. We would like to describe two of them, to illustrate the kinds of thinking involved and the variety of possible results. Both solution ideas come from the perspective of the mathematical theory of cooperative games.

In 1964, game theorists Robert Aumann and Michael Maschler proposed a general solution concept for cooperative games which is now known as the **Aumann-Maschler Bargaining Set**. (Actually, there are several different Bargaining Sets. We describe one of the simplest.) For a general elementary introduction, see [**9**]. In the 1970s this idea was applied to the spatial coalition problem ([**26**], [**20**]) and it is this application we will discuss. The Bargaining Set approach centers around the problem of a winning coalition which is considering forming: it must choose a policy point which is *fair* to all its members. In fact, any member which considers the selected point unfair will be likely to refuse to join the coalition. Thus if a potential coalition cannot find a policy point which all its members consider fair, it probably will not form.

We will illustrate the way that Aumann and Maschler make the idea of fairness operational by an example. Suppose that in the spatial situation of FIGURE 2, GHJ are considering forming a winning coalition with policy point α. Although α may seem fair (by symmetry) to us as outsiders, G thinks the policy point should be closer to him and further from J. He **objects** against J. The nature of his objection is that he can do better in another coalition:

> *G's objection against J*: "I can go to F and K and propose forming the winning coalition FGK with policy point β. That is better for me than α. Furthermore, it is also better than α for both F and K, so they would be happy to join me."

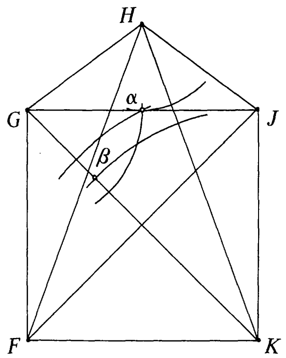


FIGURE 2. **Bargaining set theory. Point α is not a stable proposal for coalition *GHJ*.**

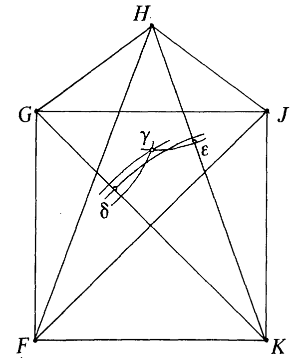


FIGURE 3. **Bargaining set theory. Point γ is a stable proposal for coalition *FHK*.**

G's objection has force, but Aumann and Maschler argue that we cannot consider α unfair, or unstable, just because one coalition member can object against it in this way. The reason is that there could then be *no* fair policy proposals. A little experimentation with the picture should convince you that *any* policy point proposed by any coalition gives rise to this kind of objection by at least one member. (Game theorists express this fact by saying that the **core** of this game is empty.) Aumann and Maschler propose that G's objection should not invalidate α if J can come up with a reasonable **counter-objection**, i.e., a convincing plan of action for what she would do if G tried to carry out the plan in his objection. This should have the following form.

J's attempted counter-objection against G: "I will form a winning coalition which includes me but not you, with a policy point which is
i) at least as good as α for everyone in my coalition, and
ii) at least as good as β for F or K, if they are involved in my coalition."

Condition ii) is necessary in order to convince G that J can compete with him for F or K if she needs them.

Unfortunately, in this case J cannot make such a counter-objection. Any point which is as close as α to J is *not* as close as β to F, and any point which is as close as α to H is *not* as close as β to K. J cannot offer enough to two parties simultaneously to convince them to join her. Since J has no convincing counter-objection to G's objection, we say that α is an **unstable** policy proposal for GHJ. In fact, GHJ has *no* policy proposal which is stable in the Aumann-Maschler sense. G could successfully object against any point to the right of α, and J could successfully object against any point to the left.

Are there coalitions which do have stable policy proposals? The answer is yes. For example, consider coalition FHK proposing policy point γ in FIGURE 3. Suppose F objects against H:

F's objection against H: "I can propose FGK with policy point δ. It is closer than γ to me, and also to G and K."

This objection can be countered.

H's counter-objection against F: "If you do, I can propose HJK with policy point ε. That's as good as γ was for me and J, and it's as good as your offer of δ for K."

In fact, H can counter any objection by F, using either GHJ or HJK. In a similar way, any member of FHK can counter any objection by any other member. The point γ (which lies on a line segment defined by inequalities which say that any objection can be countered) is a stable policy proposal for FHK.

In this example, it turns out that five three-party coalitions have Aumann-Maschler stable proposals: FHK, GJK, FHJ, GHK and FGJ. These are exactly the coalitions which "split the opposition," in the sense that the convex hull of such a coalition intersects the convex hull of its opposition (for example, the convex hull of FHK intersects the convex hull of GJ). We'll call them **internal** coalitions [**20**]. All of their stable policy points are inside the central pentagon, reasonable compromise policies which are about as good for nonmembers of the coalition as they are for members. The coalitions which do not have stable policy proposals are the coalitions like GHJ whose convex hulls are disjoint from the convex hulls of their opposition. We'll call them **external** coalitions. They are more extremist coalitions which would choose policies quite unfavorable to their nonmembers. The intuitive idea behind the instability of these coalitions is that they give their opposition so little that it is easy for a member to threaten to defect and offer the opposition more.

The prediction by the Aumann-Maschler Bargaining Set Theory is that one of the five internal coalitions (the theory doesn't say *which* one) should form, and pursue a policy represented by a stable policy point. We have seen that such a policy would always be "centrist." Do you think this always happens? Many political scientists are pleased with the kind of reasoning in the Aumann-Maschler argument. Its idea of objections being met by counter-objections seems to capture the give and take of hard bargaining, threats and counterthreats in the political arena. Nevertheless, they are uneasy with the predicted results. In the real world, external coalitions pursuing off-center policies do occur.

In 1978 political scientists Richard McKelvey, Peter Ordeshook and Mark Winer proposed an alternative solution for the spatial coalition problem which they called the **competitive solution** [**20**]. They try to capture the bargaining process in a different way. Instead of members within a coalition trying to get what is fair to them by threatening to go off and form other coalitions, McKelvey, Ordeshook and Winer think of alternative coalitions which include some common members as bidding for the support of those members. They bid by making policy proposals which are as attractive as possible to those pivotal members. A coalition can only form if it can compete successfully.

In FIGURE 4 (which is borrowed from [**20**]) the five points α, β, γ, δ and ε on the central pentagon are determined by five equidistance relationships, shown by the circular arcs. For instance, α and δ are equidistant from G. Consider potential coalition FGK offering δ and potential coalition GHJ offering α. These two coalitions are competing for G, and their proposals are equally attractive to G. Their offers to G are **balanced**. Now consider FGK offering δ and FGH offering ε. These two coalitions are competing for both F and G. The offer δ of FGK is more attractive to F, but the offer ε of FGH is more attractive to G. These offers have a kind of balance too, since neither one is more attractive to *both* parties for which the coalitions are competing. In fact, if we consider the collection of all five external coalitions GHJ, HJK, FJK, FGK and FGH with respective offers α, β, γ, δ and ε, all of these are balanced against each

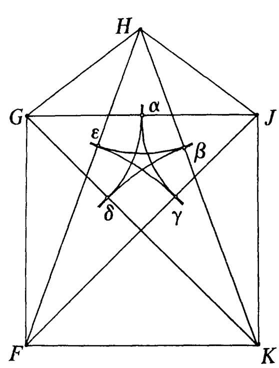


FIGURE 4. **The competitive solution.**

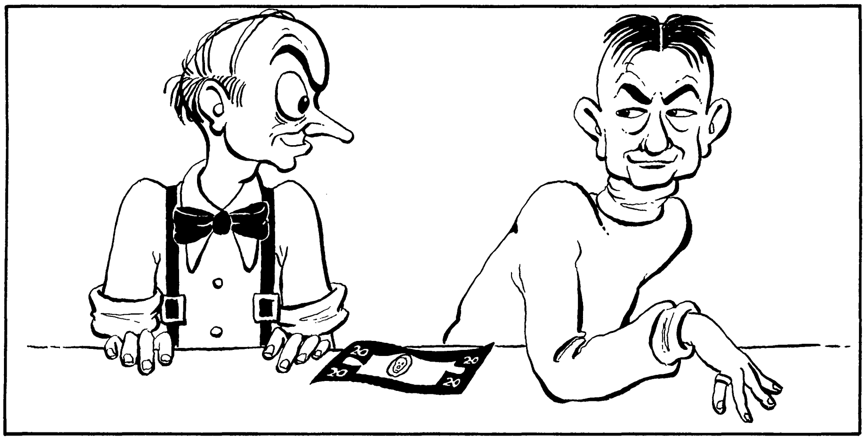
other. McKelvey, Ordeshook and Winer call a collection of coalitions and offers like this **competitively balanced.**

The five internal coalitions *FHK*, *GJK*, *FHJ*, *GHK* and *FGJ* are not represented here. Consider one of them, say *FHK*. Can *FHK* offer a proposal which would compete against all the proposals in our set of five? No, it can't. If *FHK* is to compete with *GHJ* for H, it must make a proposal which is at least as attractive to H as α. But then its offer will be less attractive than γ to both F and K, so it will lose F and K to *FJK*. The collection of five competitively balanced offers by the external coalitions cannot be invaded by other coalitional offers. Such a balanced, uninvadable collection is called a **competitive solution**. It is fairly easy to check that the five internal coalitions are not part of any competitive solution: any balanced set of proposals they might make can be disrupted by a more competitive proposal from an external coalition. McKelvey, Ordeshook and Winer predict that the coalition which forms will be one which is contained in a competitive solution. Here that means an external coalition.

This picture of coalitions trying to make offers which will hold their pivotal members is more complicated to think about than the Aumann-Maschler bargaining procedure. The problem is that a coalition is not a single entity (after all, it hasn't formed yet) but a collection of individual parties, all of whom are simultaneously offerers trying to hold pivotal members, and pivotal members seeking offers themselves. And they are doing this in many coalitions. After a while, your head begins to hurt. However, there is no question that the theory offers a prediction which is testably different from the Aumann-Maschler Bargaining Set. In the pentagon example it predicts that an external coalition will form with a policy proposal on the boundary of the central pentagon, while the Bargaining Set predicts an internal coalition with a policy point inside the central pentagon.

McKelvey, Ordeshook and Winer tested their theory by running a series of eight experimental five-person pentagon bargaining games. The subjects played for money and could win up to about $20, so they were well motivated to bargain. The results were clearcut. In all eight cases an external coalition formed, none of the policy points were inside the central pentagon, and six of the eight points were close to the competitive solution prediction [**20**]. The bargaining procedure of Aumann and Maschler is a lovely abstraction of considerable intuitive appeal, but it just isn't what people seem to do in this kind of situation. It doesn't seem to work very well for parliamentary coalitions either [**26**]. Sometimes even the nicest model has to be abandoned when reality won't cooperate.

By the way, in our original Norwegian example, the bargaining set predicts AB, AC, AD or AE, while the competitive solution predicts AB, AC or *BCDE*.



"The subjects played for money and could win up to about $20, so they were well motivated to bargain."

A dynamic model of spatial coalition formation

The complexity of the game-theoretic solutions for spatial coalition formation comes from the fact that they try to model a negotiation process in which the participants keep in mind a number of possible final results and play them off against each other. What would happen if we didn't assume such farsightedness in the parties? Remember our original assumption that each party wants to join with other parties of similar values. One short-sighted way to do this would be step by step. Go to a nearby party and offer to form a coalition with it. If it accepts, form the coalition and go together with an offer to another nearby party or coalition. Continue until your coalition is winning, or until other parties following the same strategy beat you to it.

To model this idea, we start with our parties as points in some n-dimensional Euclidean space, each party weighted by its number of seats. Consider two parties, A and B, with weights a and b, respectively, thinking about whether to form a coalition. If they do, the coalition will have to adopt a policy point, probably somewhere between A and B. We'll adopt the sociologists' "resource theory" (see the first section) and assume that the most natural policy point would be the weighted average $(aA + bB)/(a + b)$:

$$\underset{\substack{A\\(\text{weight } a)}}{\bullet}\overbrace{\rule{6em}{0.4pt}}^{\frac{b}{a+b}d(A,B)}\underset{\substack{\text{policy point}\\ \text{of } AB}}{\bullet}\overbrace{\rule{6em}{0.4pt}}^{\frac{a}{a+b}d(A,B)}\underset{\substack{B\\(\text{weight } b)}}{\bullet}$$

A would want to join with B if this weighted average policy is close to A. In fact, A would look at all possible partnerships with others and make an offer to the party whose weighted average with A would be *closest to* A. If each party does this, each party will make an offer to some other party, its preferred coalition partner. We can represent such offers as a directed graph, the **preference digraph**, in which each vertex (party) has a directed edge leading from it to its preferred coalition partner.

We will assume that two parties form a coalition if and only if their offers are reciprocal—each is the other's preferred partner. If that happens, the two original parties, A and B, say, are replaced by their coalition AB, a new actor with weight $(a + b)$ at point $(aA + bB)/(a + b)$. In this first stage, several pairs of parties may form coalitions. If none of these coalitions is of winning size, the process continues to a second stage with the new actors. As soon as a winning coalition forms, the process terminates.

Let's try this process out on the Norwegian example of FIGURE 1. First we compute A's preferred partner. The distance from A to B is 11.40. Since A's weight is 68 and B's weight is 13, the distance from A to the weighted average of A and B would be $(13/(68+13))(11.40) = 1.83$. Other distances are calculated similarly:

	B	C	D	E
Distance from A	11.40	11.18	12.37	16.03
Weighted distance from A	1.83	2.34	2.59	5.02

Since the smallest of these weighted distances from A is to B, A prefers to join with B. The complete preference digraph for stage one is shown in FIGURE 5. We invite you to verify it. Since B and D prefer each other, they join in a coalition of weight 31 at point $(13/31)(4,10) + (18/31)(7,6) = (5.74, 7.68)$. Since no coalition is yet winning, we proceed to stage two, with a new preference digraph shown in FIGURE 5. Now C and E join to form a coalition of weight 49 at $(9.16, 1.63)$. From the stage three digraph of FIGURE 5 we conclude that the winning coalition $BCDE$ will form.

In studying this model, we must first show that it always produces an answer. We must show that at every stage there is at least one pair of coalitions which prefer each other as partners, for if there were not, the process would grind to a halt. What we need to show is that any digraph

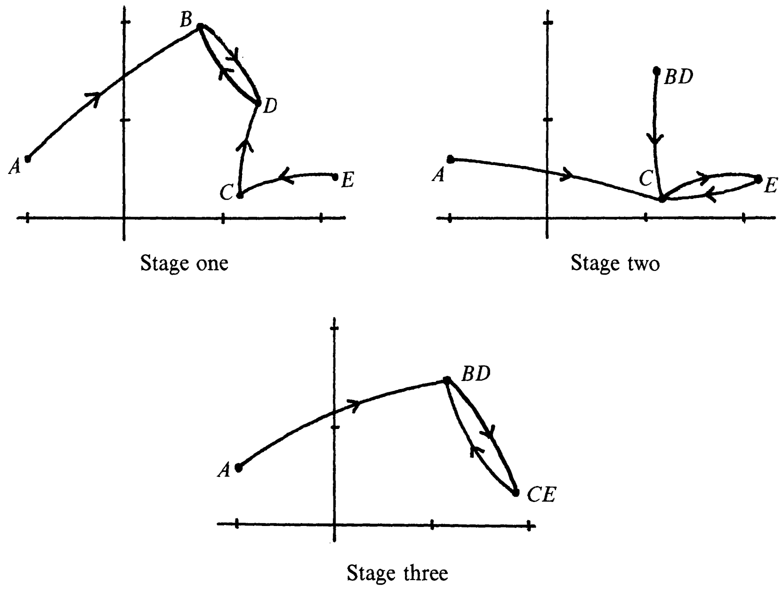


FIGURE 5. **Preference digraphs for the dynamic model.**

arising in the process must have a *cycle of length two*. A nice argument for this is in two steps.

i) *The digraph must have a cycle*. Start at any vertex. Each vertex has a directed edge leading away from it. Follow these edges in sequence. Since the number of vertices is finite, we must eventually return to a vertex we have already visited, at which point we have traversed a cycle.

ii) *No cycle can have length greater than two*. Suppose $A_1, A_2, \dots, A_n$ $(n \geqslant 3)$ were such a cycle, where A_i has weight a_i.

$$\text{Since } A_1 \text{ prefers } A_2 \text{ to } A_n, \quad \frac{a_2}{a_1 + a_2} d(A_1, A_2) < \frac{a_n}{a_n + a_1} d(A_n, A_1).$$

$$\text{Since } A_2 \text{ prefers } A_3 \text{ to } A_1, \quad \frac{a_3}{a_2 + a_3} d(A_2, A_3) < \frac{a_1}{a_1 + a_2} d(A_1, A_2).$$

$$\vdots$$

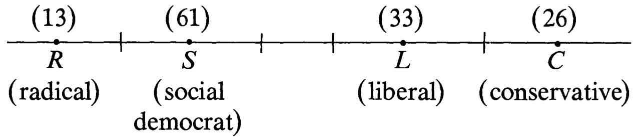
$$\text{Since } A_n \text{ prefers } A_1 \text{ to } A_{n-1}, \quad \frac{a_1}{a_n + a_1} d(A_n, A_1) < \frac{a_{n-1}}{a_{n-1} + a_n} d(A_{n-1}, A_n).$$

But if we multiply all of these inequalities together, noting that all of the terms are positive, we find that the left and right sides contain exactly the same terms, which is a contradiction.

The argument is worth contemplating. In particular, you should be sure to see how ii) depends on the condition that $n \geqslant 3$.

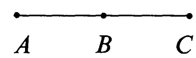
In the presentation so far, we haven't dealt with the case where some party might have a tie for its preferred coalition partner. The preference digraph would then have two directed edges from that vertex. You might like to check that the presence of ties introduces some indeterminancy into the sequence of coalition formation, but produces no serious problems. In particular, there is still always a cycle of length two at each stage.

Recall from the first section that there is some empirical evidence that governing parliamentary coalitions will be i) minimal winning and ii) connected. We should investigate how the dynamic model performs with respect to these properties. For the first, consider the following one-dimensional example, which is abstracted from the April 1953 elections in Denmark. To win, 76 votes are needed (this is larger than a majority because we have omitted some minor parties).



The dynamic process gives coalitions RS and LC in stage one. Neither wins, and in the second stage we get the grand coalition $RSLC$, which is clearly not minimal winning. The dynamic process can produce nonminimal winning coalitions. Before we get too discouraged, we might consider what happened in Denmark in 1953, which was that *no* governing coalition formed and a minority caretaker government ruled until new elections were held in September. Grofman **[15]** has proposed that if the dynamic process produces a nonminimal winning coalition, it may be a sign that it will be difficult for a governing coalition to form, or be stable if it does form. The idea is that in this case the strategic desire to expel extraneous members conflicts with primary ideological alliances.

As for connected coalitions, the dynamic process will always produce connected coalitions in a *one-dimensional* policy space. To see why, consider



and note that it will never be possible for A to prefer C *and* C to prefer A. For if the weighted average of A and C is at or to the left of B, then C will prefer B to A (since the weighted average of B and C is to the right of B). Similarly, if the weighted average of A and C is to the right of B, then A will prefer B to C. But these cases exhaust the possibilities. Hence a disconnected coalition will never form at any stage of the dynamic process.

We thought for a while that this result might be true in higher dimensions as well, but it is not. A counterexample in two dimensions is shown in FIGURE 6. In stage one, with preference digraph as shown, EF forms with weight 10 at (10,10), and AB forms with weight 10 at (2.8,0). In stage two ABC forms at $(-3.26,0)$. In stage three the winning coalition DEF forms at (10,0). The coalition DEF is not connected, since it doesn't include party A, whose policy point is in the convex hull of its members. The situation of poor A in this example is worth thinking about. Because he shortsightedly let himself be enticed away first by B and then by C, he is left out of a winning coalition which forms precisely at his preferred policy point.

We do not view the possibility of disconnected coalitions in two or higher dimensions as a defect of the dynamic model. First, the care it took to construct the counterexample hints that it is

Party	Weight	Location
A	1	(10, 0)
B	9	(2, 0)
C	9	(−10, 0)
D	10	(10, −10)
E	5	(9,10)
F	5	(11,10)
	39	

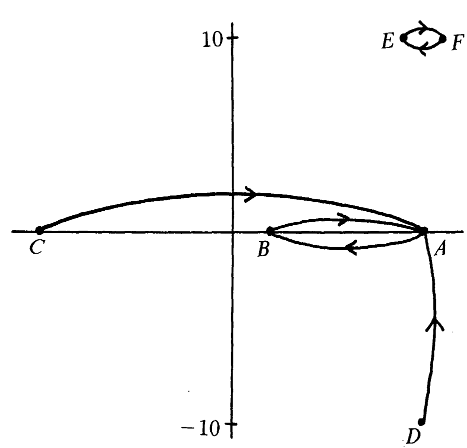


FIGURE 6. **The dynamic process forms a disconnected winning coalition.**

not easy for disconnected coalitions to arise by the dynamic process. Second, the empirical support for connected coalitions, as cited by Axelrod, for instance, comes from one-dimensional models.

The dynamic model has not yet been tested against the full range of twentieth-century data on parliamentary coalitions, but Grofman [**14**] has gathered some preliminary evidence that, for all its simplicity and shortsightedness, it predicts well the formation of governing coalitions in Norway, Denmark and Germany. For example, here are the predictions of the models we have considered for the Norwegian situation of FIGURE 1:

Theory	Prediction
Minimal winning	$AB, AC, AD, AE, BCDE$
Least resources	$BCDE$
Fewest actor	AB, AC, AD, AE
Bargaining set	AB, AC, AD, AE
Competitive solution	$AB, AC, BCDE$
Dynamic model	$BCDE$

The coalition which actually formed was $BCDE$. In Denmark in 1973 the dynamic model predicts the six-party, nonminimal winning coalition which actually formed and which is predicted by no other model. Grofman [**15**] reviews evidence that the dynamic model makes 16 correct predictions out of 18 predictions for Denmark in the period 1913–1973. This kind of performance is especially impressive since the dynamic model makes a *unique* prediction, whereas except for the least resources theory, the other theories generally do not. If you would enjoy running further tests, there is plenty of data in [**5**], [**10**], [**11**], [**13**] and [**17**].

Comparing and testing models

We have considered a number of models of coalition formation, some naive, some quite sophisticated. Some predictions are weak, others are very precise. How do we go about deciding which model is the best one? We might gather all the available data, compute the predictions of all the models, and choose the model with the highest percentage of correct predictions. Of course we would have to figure out a way to compensate for the varying degrees of precision with which the models make their predictions (see [**27**] for a statistical technique to do this). However, in the domain of social science, things are not generally that easy. There are at least two factors which make the decision among competing models a much more subtle process.

"We... choose the model with the highest percentage of correct predictions."

The first factor involves the nature of data in social science. It is hardly ever just *there*. It has to be carefully gathered, organized and put in suitable form before it can be used to test models. In spatial coalition theory we see this problem in a particularly basic form. Recall that we have assumed as given that parties are located as points in an n-dimensional ideological space. In the literature, the methods used to get such a placement have varied from individual "expert judgment" through quantitative analysis based on public perceptions or previous appearances in coalitions. At the very least, the inexactness of placement methods should caution us against claiming precision in prediction. There are a number of additional problems, for instance

i) there may be, and are, inconsistencies in the spatial placements obtained by different analysts using the same or different methods;
ii) although Euclidean distance has usually been used as the measure of ideological proximity, other metrics (for instance the "taxicab metric" $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$) might have an equal or better claim to validity;
iii) there may be some logical circularity in deriving likely coalitions from spatial placements, since parties might be perceived as similar *because* they have often been in coalitions together.

With enough caution, sensitivity analysis and modesty for claimed results, we think these problems can be controlled, but they illustrate the subtle relationship between data and models in the social sciences.

The second factor complicating the process of deciding among coalition models is that we are trying to understand a process which is so complicated that it probably does not make sense to ask for a unique *best* model. Some of the models have parties concerned mainly with power, some mainly with ideology. The game theory models posit careful rational consideration of alternatives; the dynamic model relies on more shortsighted behavior. We will likely find that different models work best in different situations. For instance, the political cultures of various countries may result in parties balancing differently concerns for power and ideology. Some systems may be conducive to hard rational bargaining of the kind that McKelvey, Ordeshook and Winer's experimental subjects were motivated and able to do, and the competitive solution may be applicable. Other systems may make flexible offers and counteroffers harder to make, and the step-by-step process of the dynamic model will be more suitable. In fact, if different models predict well for different cultures, that could indicate interesting things about differing motivations and concepts of bargaining. This kind of comparative richness would be lost if we insisted on one best model.

In general, the role of mathematical models in the social sciences is more flexible than in the physical sciences. We rarely expect that a model will embody the "correct" understanding of a phenomenon, only that it will give insight, lead to new and interesting questions and directions for further study. Models are not an end of thought, but an aid to thought. If you would like to look at other ways in which mathematical models are being used in political science, we would recommend **[4]**, **[21]**, **[24]** and **[1]**.

Coda: Applying the dynamic model outside of politics

We would like to close by illustrating one way in which the dynamic model of coalition formation might be useful outside of political science. Consider the geographical problem of the formation of a trading network among spatially separated communities. We think of the communities starting off isolated. Each community decides on a preferred trading partner, which in the simplest case might be just the *closest* other community. Two communities form a **trade link** between them if and only if they are each other's preferred trading partner. Once a trade link is established, communities connected by it are considered together as a **trading coalition**, and the

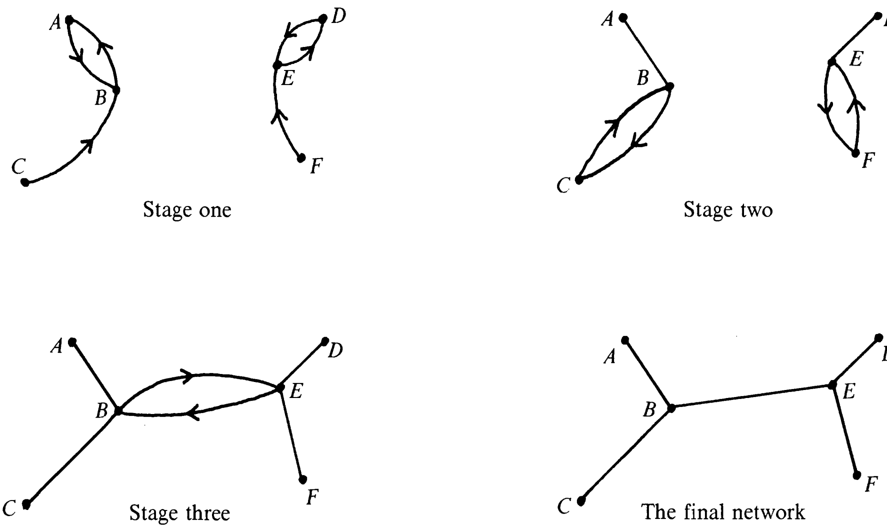


FIGURE 7. **The dynamic model of the formation of a trading network.**

distance between two trading coalitions is simply the distance between their closest members. The process continues until all of the communities are connected by a trading network. FIGURE 7 shows an example of the process. Notice the changes we have made from the political model. First, the communities don't amalgamate at their center of gravity, but stay fixed and are joined by a trade link. Second, the distance between coalitions is measured from the closest members, and that is where the trade link is drawn. For this modification, the proof that at least one trade link must form at each stage is trivial: the two coalitions which are closest together must prefer each other.

In this model, two things might be of interest to geographers—the sequence in which trade routes are built up over time (could we observe it?) and the final trading network. For the latter, a little thought should convince you of the following:

i) the final network will always be a **tree** (a connected graph with no cycles) which spans the communities, and
ii) this tree is the same tree as would be obtained if you started by drawing the shortest possible edge, and at each step added the next shortest edge which would not complete a cycle. (The order in which the edges come in may be different, but the final result will be the same.)

Now it is well known that the algorithm in ii) generates the **minimal spanning tree** of the communities, i.e., the tree which has the smallest total length among all possible spanning trees (see [**7**] or [**22**]). We thus have the result that communities which shortsightedly build up a trading network by this dynamic process will efficiently solve the global problem of joining themselves by a minimal length trading network.

Grofman and Landa [**16**] have used this model to investigate the development of the "Kula ring" trading network in the Melanesian Islands, which was first described by Malinowski in 1922.

One useful modification of the model might be to weight communities by their value as trading partners if distance were not a consideration. We might, for instance, weight them by the wealth of their economies. A community A might then judge the attractiveness of another community B by $b/d(A,B)$, where b is the economic weight of B. A trading link would be established if two communities were most attractive to each other, and the weight of a coalition would be the sum of the weights of its members. For this modification, we would once again have to prove that at each stage there is a pair of mutually most attractive coalitions. Fortunately, the argument in the third section goes through *verbatim*, except that the inequalities which must be multipled together now have the form

$$\frac{a_{i+1}}{d(A_i, A_{i+1})} > \frac{a_{i-1}}{d(A_{i-1}, A_i)}.$$

In fact, the argument will work when the attractiveness of A_j for A_i is measured by a wide variety of functions of a_i, a_j and $d(A_i, A_j)$. Modifications of the dynamic model of coalition formation might apply in many cases where political, economic or social agents join with other agents whose attractiveness as partners depends on their "weight" and their "location."

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