ARTICLES

Which Tetrahedra Fill Space?

Early mathematicians gave some puzzling answers; today the problem is not yet completely solved.

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Filling space by fitting congruent polyhedra together without any gaps is one of the oldest and most difficult of geometric problems, and has a fascinating history. It arose first in ancient times in relation to Plato's theory of matter; during the subsequent 2300 years of its development, it has continued to receive its principal stimulus from physicists and others interested in the structure of the solid state. In its most intuitive form, the problem is that of determining the shapes of building blocks—the building blocks of architecture, of inorganic and organic matter, of space itself. Its origin can be traced to Plato's atomic theory: the hypothesis that all matter is the result of combinations and permutations of a few basic polyhedral units. The mathematical question is: *what shape must such a unit have if it is possible to fill space without gaps by figures congruent to that single unit*? This simply stated geometric problem is still unsolved, despite considerable efforts devoted to it over the ages.

That rectangular solids or, more generally, parallelopipeds can be fitted together to fill space was known to the earliest bricklayers, but that any other polyhedra have this property is less obvious. Plato, as we shall see, assumed the existence of such polyhedra, but Aristotle was the first to get down to details. In the process he made a mistake that generated a controversy lasting nearly 2,000 years.

Aristotle asserted that, of the five regular solids (FIGURE 1), not only the cube but also the tetrahedron fills space. That this is incorrect (FIGURE 2) does not seem to have been evident at that time, and many of the later Aristotelian scholars—if they realized that something was amiss—apparently assumed that somehow *they* must be mistaken. In trying to justify Aristotle's erroneous assertion, they raised the interesting question of which tetrahedra actually do fill space, and they developed some of the techniques used today in the study of space-filling polyhedra.

The early history of the space-filling problem was discussed in detail by Dirk Struik in 1925 [**1**]; there he showed how Aristotle's error, for all the confusion it caused, indirectly played a constructive role in the development of the theory of polyhedral angles. The story is instructive in many ways. It shows how errors can arise through misunderstanding of a problem or excessive deference to a great thinker, and how they can be perpetuated for these reasons or through simple carelessness, sometimes even after the problem has been properly resolved. The first section of this paper is based on Struik's article. In the second, we briefly sketch the important role that the problem has played in the development of the theory of the structure of crystals from about 1600 to the present. In this discussion we hope to show that the interaction between geometry and natural sciences can be profitable for both sides. Finally, we discuss the question inadvertently raised by Aristotle: *which tetrahedra fill space and which do not*?

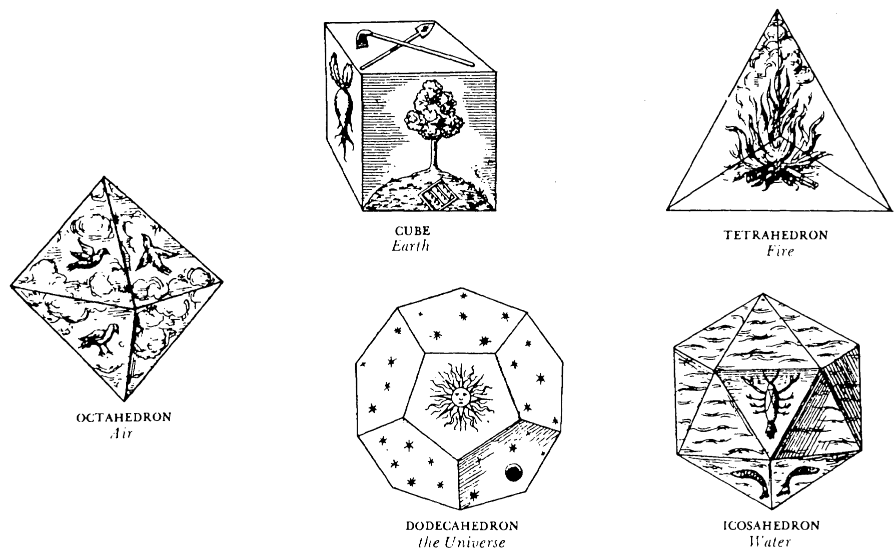


FIGURE 1. **The five regular solids as depicted by Johannes Kepler in *Harmonices Mundi, Book II* (1619). Redrawn by John Kyrk, this is Illustration 1 in [3]. (Reprinted with permission.)**

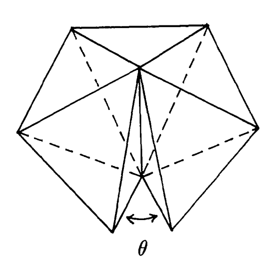
§1. An error for almost 2000 years

The discovery of the regular solids, and the proof that there are exactly five of them, was one of the great mathematical achievements of the ancient Greeks. They were discussed in detail by Euclid in the final book (XIII) of the *Elements*; it has even been suggested that the purpose of the *Elements* was to provide a rigorous treatment of their construction. Plato seems to have been the first to "apply" the theory of these polyhedra in the interpretation of nature: they were the basis of his theory of matter, which is presented in his dialogue *Timaeus*. According to Plato, all matter consists of combinations of four basic "elements": earth, air, fire, and water (this corresponds rather well to our present concept of the phases of matter). The elements of each type are composed of particles, "far too small to be visible," of definite shape: the earth particles are cubes, the water particles regular icosahedra, the air particles regular octahedra, and those of fire, regular tetrahedra. (The fifth regular solid, the pentagonal dodecahedron, was associated with the cosmos.) The varieties of the elements (such as different kinds of stone, or different liquids) were explained on the hypothesis that the basic particles come in many different sizes, while substances which are mixtures of elements were assumed to consist of mixtures of the corresponding particles.

Aristotle argued that Plato's theory is incompatible with reality. If a substance is composed of particles of a given shape and size, then these particles must pack together to fill the space

FIGURE 2.

The regular tetrahedron does not fill space without gaps. Its four faces are equilateral triangles, from which it follows that its dihedral angles α (the angles between adjacent faces) are equal to arccos $(1/3)$, or $\alpha \approx 70^\circ\ 32'$. If 5 tetrahedra are fitted around an edge, there is a gap whose angular measure θ is less than α, and we conclude that regular tetrahedra do not fill space when arranged face-to-face. In any other arrangement a dihedral angle of $\pi - \alpha$ is created, which cannot be filled by regular tetrahedra.



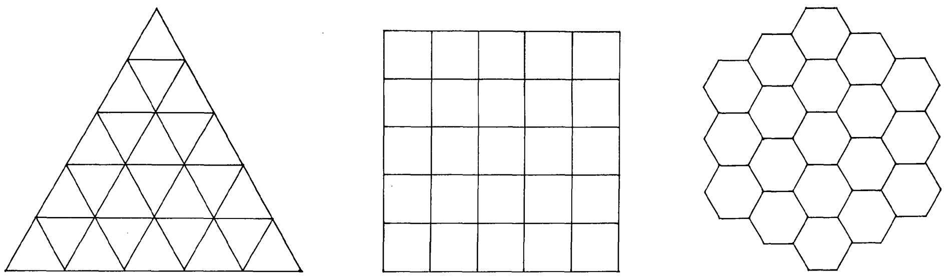


FIGURE 3. **The plane can be filled with squares, equilateral triangles, and regular hexagons. No other regular polygons can fill the plane without gaps.**

occupied by the substance, that is, they must fill space without leaving any gaps. A gap would mean empty space, or a vacuum, which according to the Aristotelian theory of motion cannot occur in nature. But some of the regular solids do not fill space. Thus, remarked Aristotle, "In general it is incorrect to give a form to each of the singular bodies, in the first place, because they will not succeed in filling the whole. It is agreed that there exist only three plane figures that can fill a place, the triangle, the quadrilateral, and the hexagon, and only two solid bodies, the pyramid and the cube. But the theory demands more than these, because the elements they represent are greater in number" (quoted from *De Caelo* III, 306b). This was considered to be a serious argument against the ancient atomic theory, which consequently became increasingly unpopular.

We may assume that Aristotle was referring to the fact that the only *regular* polygons which fill the plane with copies of themselves are the square, the equilateral triangle, and the regular hexagon (FIGURE 3). From this and from the context of his remark, we conclude that Aristotle believed that the regular octahedron and icosahedron do not fill space (in this he was correct) while the cube and regular tetrahedron do. He gave no evidence for his claim. Struik remarks, "This passage, which is only reported incidentally in a modern investigation of Aristotle's mathematics, caused the ancient writers considerable concern." Thus we find a series of commentators on Aristotle discussing the number of tetrahedra that can "fill the space about a point," that is, be packed together so as to share a vertex. Simplicius, a scholar and commentator who lived in the first half of the sixth century A.D., asserted that the number of such tetrahedra is twelve, but gave no reason. He also stated that Potaman (a philosopher who probably lived in the first century A.D.) had concluded that the number was eight, by the following reasoning. The maximum number of cubes which can share a vertex is eight. If we truncate each of the cubes meeting at that vertex, we obtain eight tetrahedra which fill space about a point (FIGURE 4).

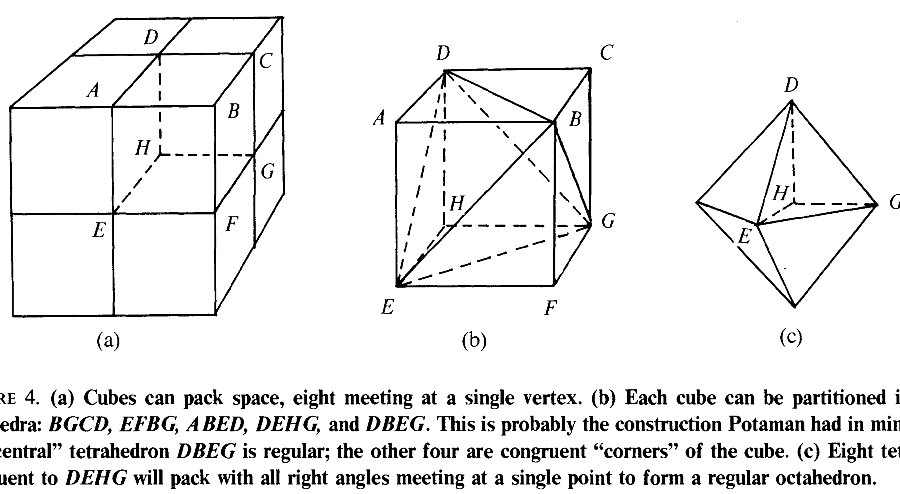


FIGURE 4. **(a) Cubes can pack space, eight meeting at a single vertex. (b) Each cube can be partitioned into five tetrahedra: *BGCD, EFBG, ABED, DEHG,* and *DBEG*. This is probably the construction Potaman had in mind. Only the "central" tetrahedron *DBEG* is regular; the other four are congruent "corners" of the cube. (c) Eight tetrahedra congruent to *DEHG* will pack with all right angles meeting at a single point to form a regular octahedron.**

Potaman's argument, as reported here, has some defects which illustrate the sort of difficulties which have plagued this problem throughout its history. In the first place, the eight tetrahedra Potaman obtains by truncating eight packed cubes are not regular: the tetrahedral faces which meet at the vertices of the cube are right, not equilateral, triangles. While Aristotle did not state explicitly that he meant regular tetrahedra, it is, as we have seen, reasonable to assume that this is what he intended. Even if we ignore this objection, another remains: filling the space about a point is *not* the same thing as filling space as a whole. Some packing arrangements cannot be continued to fill all of space. It is possible that Potaman thought that if his "truncation" procedure was carried out on all the cubes in a regular packing of cubes, then each cube would be partitioned into congruent tetrahedra; if this were so, then the tetrahedra would be space-fillers. While his construction does dissect each cube into five tetrahedra, these five tetrahedra are not congruent: the "central" tetrahedron is regular. Also an argument similar to that in the caption to FIGURE 2 shows that his "vertex tetrahedra" do not fill all space. Nevertheless, Potaman's technique, partitioning a known space-filler into congruent parts, is one of the most useful we have for constructing new space-filling polyhedra.

Aristotle's error not only stimulated the development of this technique; it also led to the earliest attempts to define the measure of a polyhedral angle. The 12th century Arabic commentator on Aristotle, Averroës, seeking to justify Aristotle's remark, developed a theory of angle measure which beautifully served its purpose. According to Averroës, the measure of a trihedral angle (such as the angle at the vertex of a cube or of a tetrahedron) is the sum of the face angles that form it. Thus the measure of a vertex angle of a cube is 270°, because each of the three face angles has measure 90°, and the measure of a vertex angle of a regular tetrahedron is 180°, because three 60° angles sum to 180°. Now, he reasoned, since eight cubes fill the space about a point, a necessary and sufficient condition for space-filling by tetrahedra is that the sum of the trihedral angles meeting at a point be equal to the product $8 \times 270°$. Since $12 \times 180 = 8 \times 270$, it follows that twelve regular tetrahedra fill the space about a point, in agreement with Simplicius.

Averroës's theory of angle measure was generalized by the 13th century English Franciscan scholar Roger Bacon to include polyhedral angles formed by four or more planes. In the course of this, Bacon discovered that Aristotle had missed something: the measure of a vertex angle of an octahedron is $4 \times 60° = 240°$, and $9 \times 240 = 8 \times 270$, whence nine octahedra fill space about a point! Bacon regarded this result as a significant advance. But he was aware that there was some controversy about this question since in Paris "a fool had asserted in public" that twenty pyramids fill the space about a point. Bacon added that to settle the matter it would be necessary to understand Euclid's Book XIII. But, Struik says, "It is indicative of what men in that time could and could not do, that they preferred lengthy disputes to either calculating according to Euclid or taking the trouble to construct a single model."

Actually it is intuitively reasonable that the measure of a polyhedral angle should be, in some way, closely related to the sum of the face angles comprising it, but the relationship is not as simple as these scholars supposed (see below). That something was wrong with the Averroës-Bacon theory was first pointed out by the English scholastic Thomas Bradwardinus (1295-1349): if twelve regular tetrahedra filled the space about a point, then they would together form a convex polyhedron with twelve equilateral triangular faces, which would be a sixth regular solid. Also, he noted, the theory must be incorrect because Aristotle did not include the octahedron in his list of space-fillers. According to Bradwardinus, Aristotle's tetrahedra could be obtained by joining the vertices of a regular icosahedron to its center; in this way we obtain the twenty tetrahedra of the Parisian fool. Bradwardinus was unsure whether these tetrahedra were regular. (They are not, but unlike Potaman's tetrahedra this is not obvious: detailed calculations are needed to establish the fact that the ratio of the length of the icosahedral edge to the vertex-center distance is approximately 1.05.) Perhaps Bradwardinus, like some later commentators, would have accepted the nonregularity of these tetrahedra on the grounds that Aristotle did not explicitly require regularity. He appears not to have inquired whether this packing arrangement could be repeated to fill all of space (it cannot, since the icosahedron is not a space-filler).

Bradwardinus's refutation of the Averroës-Bacon theory was a significant achievement, but it should be pointed out that it is not strictly correct. The regular solids are characterized by the additional requirement that the same number of faces meet at each vertex. If two regular tetrahedra are juxtaposed along a face, we obtain a polyhedron whose six faces are equilateral triangles, but it is not a regular solid. There are four other "irregular" convex polyhedra (sometimes called "deltahedra" [3]) all of whose faces are equilateral triangles, including one with twelve faces.

Bradwardinus's argument could have been extended to prove more rigorously that nine octahedra do not fill the space about a point. For if we cut the octahedra by planes passing through the endpoints of the edges meeting at the point, we obtain a convex polyhedron with nine square faces, which is impossible. This suggests another way of deciding whether or not a given arrangement of polyhedra fills the space about a point: find the shortest polyhedral edge e which meets the point, then draw a sphere about the point, choosing the radius r of the sphere to satisfy $r \leqslant e$. In this way we obtain a partition of the sphere into spherical polygonal regions whose edges are the traces on the sphere of the polyhedral faces which meet at the point, and whose vertices are the points at which the sphere cuts the edges which bound them. Later, when spherical trigonometry had been developed, the measure of a polyhedral angle was correctly defined to be the area of the spherical polygon found in this way. Thus a necessary and sufficient condition for the polyhedra to fill the space about a point is that the sum of the areas of these polygons equal the surface area of the sphere.

Only in the 15th century, when Euclid was again studied, did the confusion begin to be resolved. Johannes Müller, or Regiomontanus (1436–1476), the author of an important work on spherical trigonometry, was the first to discuss the problem in a critical spirit, as we can tell from the lengthy title of his manuscript, "On the five like-sided bodies, that are usually called regular, and which of them fill their natural place, and which do not, in contradiction to the commentator on Aristotle, Averroës." Unfortunately this work was lost, but subsequent authors, probably influenced by him, discussed the problem in a similar way, pointing out that it is clear from Euclid's construction of the icosahedron that the tetrahedra obtained from it are not regular. They also noted that together six regular octahedra and eight regular tetrahedra fill the space about a point (this is implicit in Potaman's construction: the bases of the eight tetrahedra meeting at a cube vertex together form a regular octahedron; see FIGURE 4). Except for the cube and combinations of the tetrahedra-octahedra packing with the cube, there are no other ways to fill space with regular solids.

Despite this criticism, Aristotle's error in its various guises persisted for a long time afterwards; scientists who should have known better perpetrated it by carelessly accepting the earlier fallacious arguments. Even when the error was finally generally admitted, some scholars continued to defend Aristotle on the grounds that he had not explicitly required regularity. The correct formula for the area of a spherical polygon was first published in a book by Albert Girard, in 1629. Later, the Polish mathematician J. Broscius (1591-1652) devoted a large portion of an important book to a thorough discussion of this question. In the course of his argument he developed a formula for the area of a spherical polygon and it is for this that the book is best known today. Here at last the problem of filling space about a point was discussed correctly and in detail.

This achievement came at the time when science was turning from speculation to experiment. The structure of matter again became a focus of interest. In the study of crystals, the problem of filling space with congruent polyhedra took on a new significance.

§2. Crystallographers revive the problem

Plato had not been concerned with the problem of how *external form* is achieved by the stacking of particles. It is in the sixteenth and seventeenth centuries that we find the first investigations of this issue. The great astronomer Johannes Kepler (1571-1630), who also made an important contribution to the problem of filling the plane with polygons, became interested in the

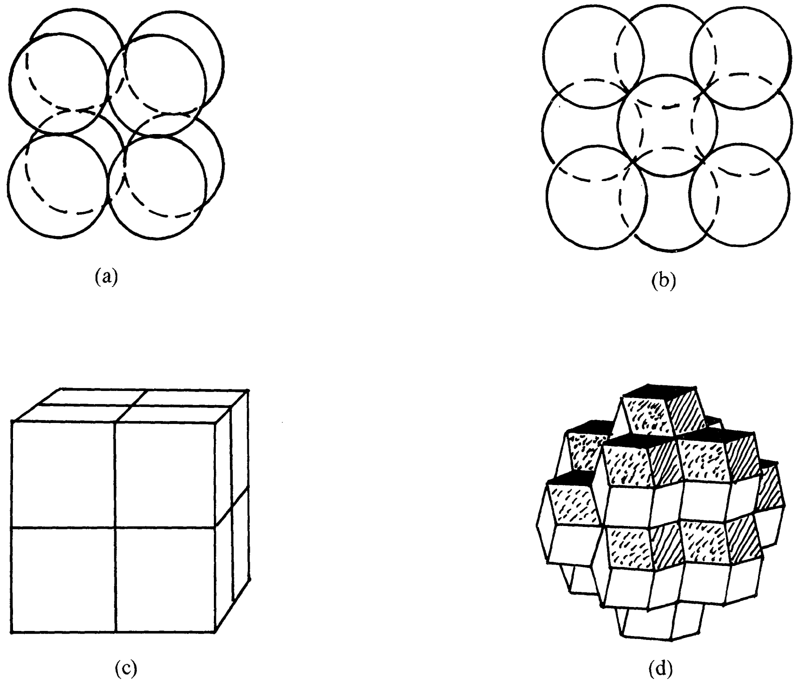


FIGURE 5. **(a) In simple cubic packing, spheres are arranged at the vertices of cubes; each sphere touches six others. (b) In face-centered cubic packing, spheres are arranged at the vertices and face centers of cubes; each sphere touches twelve others. (c) When packing (a) is compressed, the spheres are deformed into polyhedra with six faces (cubes). (d) When packing (b) is compressed, the spheres are deformed into polyhedra with twelve faces (rhombic dodecahedra).**

causes of the forms of snowflakes and wrote a booklet about his ideas as a New Year's gift to a friend in 1611 [9]. Postulating that snowflakes are composed of minute spheres of ice, he studied various packing arrangements for spheres and also some of the polyhedral forms that spheres would assume if the packing arrangements were uniformly compressed. In this way, he discovered several space-filling polyhedra. For example, if the spheres are arranged in what is known as simple cubic packing, the compression forms are cubes; if they are arranged in the so-called face-centered cubic packing, the compression forms are rhombic dodecahedra (FIGURE 5). It is important to note that in Kepler's work on snowflakes we find a completely new approach to the space-filling problem. All of the authors whose work was described in the preceding section were principally concerned with fitting given polyhedra together locally—specifically, matching faces at a vertex. (Kepler also took this approach, in his study of plane tilings.) The properties of the spatial patterns which could be generated by extending these local packings do not seem to have been considered important (indeed, as we have seen, some authors did not even investigate whether an extended pattern existed). The compression polyhedra Kepler obtained from sphere-packings, however, were solutions of a global problem: what kinds of polyhedra pack together according to the requirements of a given repeating pattern?

By this time the atomic hypothesis had been revived and was being vigorously debated. Sphere-packing was a popular approach to the study of matter. In 1665 the English scientist Robert Hooke stated that he could show that all crystalline forms could be explained by a few basic packing arrangements of spherical atoms ("had I time and opportunity") and gave several examples. But spheres, even when packed together as closely as possible, still leave gaps; the vacuum problem was a persistent difficulty (although the existence of a vacuum had been demonstrated in 1643). One way to get around it was to assume that atoms are not spherical but polyhedral in form. The first post-Platonic theory of this sort was that of the Italian physician and mathematician Domenico Guglielmini (1655–1710) who was interested in the structure of salts. There are, he said, four principal types of salts, and the atoms of each have the form of a polyhedron: a cube, a hexagonal prism, a rhombohedron, or an octahedron. The basic salts are constructed of atoms of a single shape; other salts are formed by combinations of these atoms. We see here that the theory of polyhedral atoms, which was attractive for many reasons, did not solve the problem it was intended to solve, since (regular) octahedra do not fill space—as Guglielmini was aware.

Both the sphere-packing and polyhedra-packing theories were based on the assumption that the external geometry of crystals is the result of some sort of structural regularity, despite the fact that the forms of the crystals of a given mineral species can vary greatly (FIGURE 6). For many years this view was as controversial as the atomic hypothesis on which it was based. In about 1782, however, it was discovered empirically that there is a definite relationship between the various polyhedral forms assumed by the crystals of a species. This strongly suggests that the external form of crystals is a reflection of something fundamental and characteristic.

This relationship is known to crystallographers as the Law of Constancy of Interfacial Angles; it was stated in its most general form in 1783 by the French mineralogist J. B. L. Romé de l'Isle (1736–1790). It can be described in the following way. The faces of a polyhedron can be grouped together in families which are related by symmetry. As we saw in FIGURE 6, the faces of a given family may be larger or smaller in one individual crystal than in another, or even entirely absent, but in every case the interfacial angles are exactly the same, and are characteristic of the species. Romé pointed out that for each crystal species, the families of faces can be obtained from a single basic form by truncating its vertices and edges, and devoted a book to demonstrating this. (Truncation still seems to hold a tremendous fascination for many people.)

Romé himself believed that speculation about the structure of crystals was premature and so the study of the implications of his work was left to others. Not long afterward the French abbé and mineralogist Rene Just Haüy (1743–1827) revived and expanded the polyhedral theory of crystal structure; his work marks the beginning of the modern science of crystallography.

We will note here only the chief geometrical features of Haüy's theory. The building blocks of crystals are polyhedra which can be regarded as "crystal molecules" (which are not necessarily the same units as chemical molecules). These polyhedra may be parallelopipeds, tetrahedra, or triangular prisms: the type is characteristic for a crystal species. The "nucleus" of a crystal consists of several of these polyhedra grouped together, and the crystal grows by the accretion of

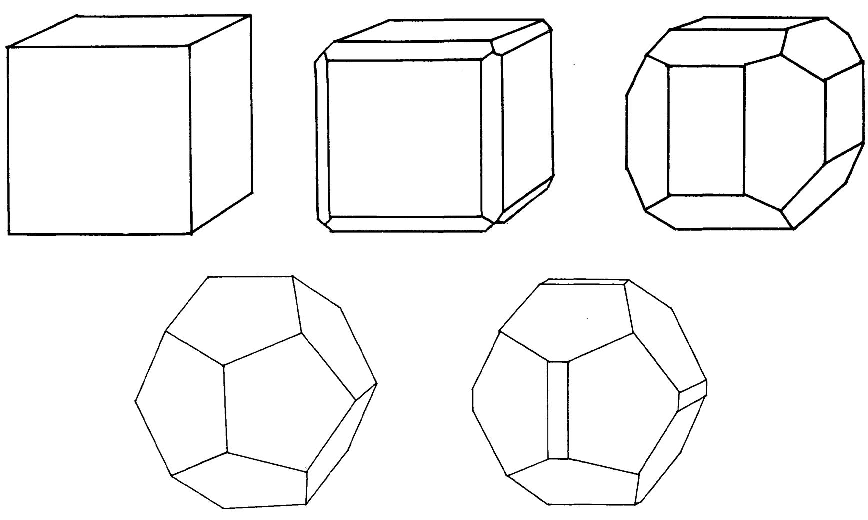


FIGURE 6. **Adapted from Goldschmidt's *Atlas der Kristallformen.* Five crystals of pyrite. The dodecahedron at lower left is not regular.**

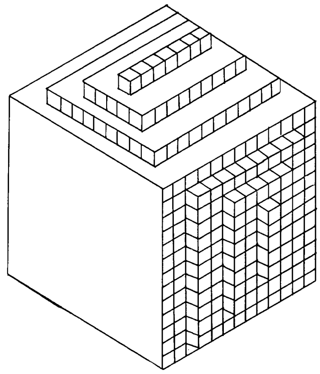


FIGURE 7. **Haüy's construction of pyrite from parallelopiped blocks.**

more of them, now grouped together to form parallelopipeds (FIGURE 7). Some crystal faces are thus smooth and others stepped, but the latter appear smooth to us because of the submicroscopic size of the steps. On this hypothesis Haüy was able to explain the forms that crystals assume, and why other forms (such as the regular icosahedron and pentagonal dodecahedron) cannot exist in the crystal world. He was also able to give an explanation of certain physical properties of crystals, such as cleavage.

Nevertheless, Haüy's theory was controversial. His constructions did not always agree very well with measurement, which provoked severe criticism of his work. The arguments centered on the reality of his building blocks. Only later was their value as an abstract model understood. Haüy's work led to the modern concept that periodicity—regular repetition in all directions—is the fundamental structural characteristic of crystals.

The first steps toward this abstraction were taken by the German physicist Ludwig Seeber in 1824. Pointing out that Haüy's theory could not explain the expansion and contraction of crystals with changes of temperature, he proposed replacing the parallelopiped blocks by a system of points representing their centers of gravity, which he called the **space lattice**. This model has been exceptionally fruitful.

Lattices can differ in their symmetry and in the way in which the points are arranged. (In 1849 August Bravais proved that there are exactly fourteen types.) The points of a space lattice can be represented by the endpoints of vectors of the form $\vec{v}=\vec{a}x+\vec{b}y+\vec{c}z$, where $\vec{a}$, $\vec{b}$, and $\vec{c}$ are three linearly independent vectors and x, y, z range over the integers. If we calculate $|\vec{v}|^2$ we obtain a homogeneous quadratic form in three variables. Conversely, each such form represents a space lattice. But different forms can represent the same lattice: how can we tell which forms are equivalent? This question is closely related to the number-theoretic problem of the "reduction" of quadratic forms, that is, the identification of the forms by the parameters $\vec{a}, \vec{b}, \vec{c}$. It is of interest to us because a major advance in the space-filling problem came indirectly from the reduction problem, of which Seeber himself was the first to find a solution (1830). Seeber's work was correct but aesthetically unappealing: it appeared to be unnecessarily long and complicated. This prompted efforts by several mathematicians to simplify it, and it was in the course of this that P. Dirichlet introduced a construction for the region of space closer to a given lattice point than to any other (FIGURE 8). If the **Dirichlet region** is constructed for each point of a space lattice, we obtain a filling of space by congruent convex polyhedra whose centers lie at lattice points.

The polyhedra which form Dirichlet regions of a space lattice are related to one another by translation. In general, polyhedra which lead to space fillings by translation are called **parallelohedra**. Parallelohedra are the building blocks of space lattices and thus are clearly important for theoretical crystallography. But for many years there were controversies about their physical interpretation. Is crystal structure really periodic? If so, is the block structure really an appropriate model? How are the blocks related to chemical structure? And so forth.

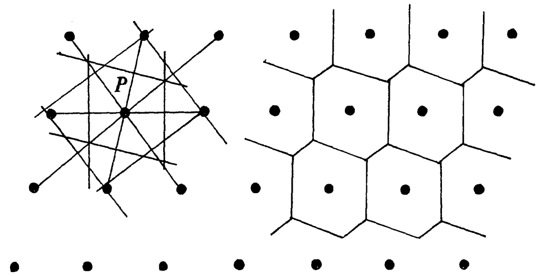


FIGURE 8. **Given a discrete point set, the Dirichlet region of a given point *P* is the region of space closer to *P* than to any other point of the set. It can be constructed by joining *P* to each of the other points by straight line segments, bisecting the segments, and finding the smallest convex region bounded by the bisectors. The Dirichlet regions of a space lattice are congruent polyhedra in parallel orientation which fit together face-to-face to fill space without gaps. (Here the Dirichlet regions of some of the points of a plane lattice are shown.)**

The great Russian crystallographer and geometer E. S. Fedorov (1853–1919) believed that the parallelohedra into which a crystal can be partitioned contain groupings of chemical molecules, and that by partitioning the parallelohedra in turn into congruent regions (Potaman's principle) we obtain the true subunits of the crystal. Fedorov investigated parallelohedra in detail, and proved the remarkable theorem that the convex parallelohedra can be classified into five topological types: the cube, the hexagonal prism, the rhombic dodecahedron, a dodecahedron with eight rhombic and four hexagonal faces, and the truncated octahedron (FIGURE 9). This was the first general result in the theory of space-filling polyhedra and is still the most important one. Fedorov included this theorem, along with many other interesting results, in a book, *An Introductory Study of Figures*, which he had begun at the age of 16. It was finally published in 1885, after being turned down by several mathematicians. (P. Chebyshev refused even to read it, remarking that "contemporary science is not interested in these questions.") Fedorov's book has never been translated into a western language, but a simplified proof of his theorem is contained in [7].

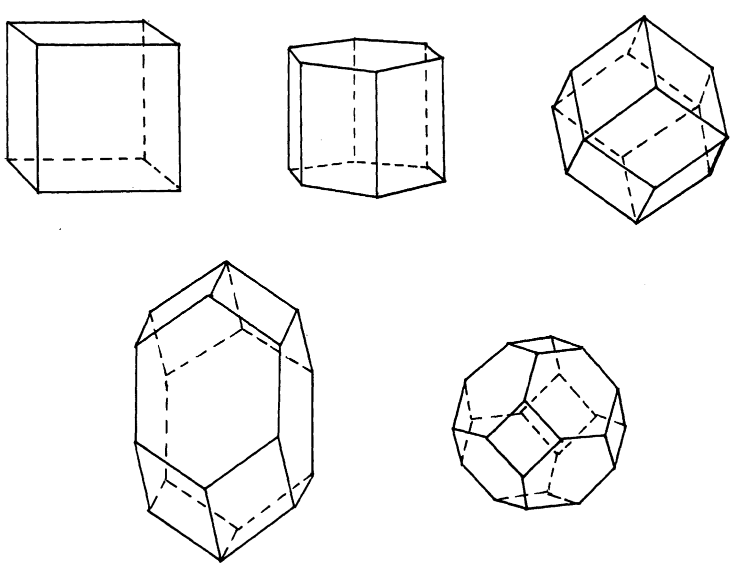


FIGURE 9. **Fedorov's five topological types of parallelohedra.**

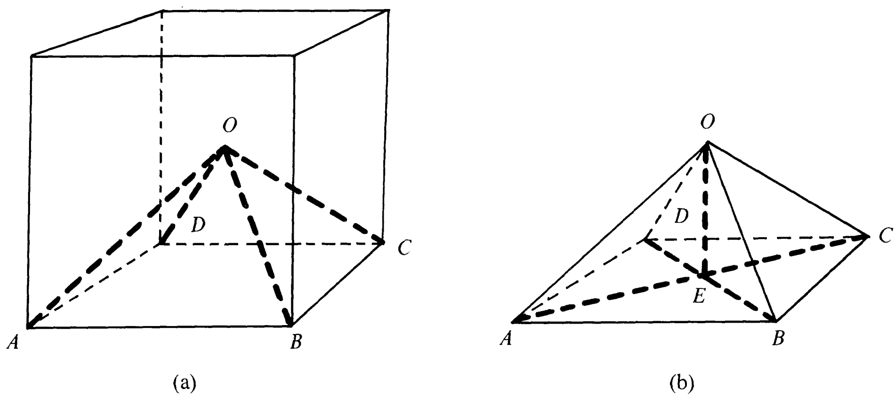


FIGURE 10. **The cube can be partitioned into twenty-four congruent tetrahedra. Let E be the center of the face $ABCD$ and O the center of the cube. (a) Joining A, B, C, and D to O forms a pyramid. (b) Joining A, B, C, D, and O to E partitions the pyramid into four congruent tetrahedra. Repeating this construction for each of the cube faces, we obtain the remaining twenty congruent tetrahedra.**

In 1912 the space-lattice theory of crystals was validated by x-ray techniques. Although Fedorov's view of the molecular groupings in crystals turned out to be incorrect, the space-filling model continues to provide useful interpretations of crystal structures. Thus the space-filling problem is a subject of active research by crystallographers as well as by mathematicians. Dirichlet regions are of particular interest; with the aid of computers they can now be constructed for complicated sets of points. Some of these regions are tetrahedra, and so provide some answers to our title question.

§3. Which tetrahedra fill space?

We are back to our title question, and in this section examine what answers are known. The techniques used to search for answers have their origins in the history we have outlined in the previous sections. Let us begin with an example reminiscent of Potaman's technique: a dissection of a cube—but into congruent tetrahedra. If we join each vertex of a cube to its center, it is dissected into six congruent pyramids. Each pyramid can be further dissected into four congruent tetrahedra by joining each of its vertices to the center of its square face. Thus we obtain a partition of the cube into twenty-four congruent tetrahedra (FIGURE 10). Other space-filling tetrahedra can be found by further partitioning this one (FIGURE 11). In the search for space-filling polyhedra, it seems logical to begin searching for such tetrahedra, since the smallest number of faces are involved. But even today, this problem is not solved, nor has a comprehensive technique to discover and enumerate all such tetrahedra been developed.

It is important to note that there is no a priori reason why space-filling tetrahedra must satisfy either the local requirements imposed by the authors discussed in Section 1 or the global requirements of those discussed in Section 2. We shall see that there are space-filling tetrahedra which do not pack together along whole faces; it is possible that there are even space-filling arrangements in which the tetrahedra are not grouped together into the units of a repeating pattern. However, the imposition of local or global requirements (or both) makes the space-filling problem mathematically tractable (to some extent). Potaman's principle, Bradwardinus's technique, and Dirichlet's construction are still the principal tools available at the present time.

Our discussion will be simplified if we first consider the various ways in which a tetrahedron can be partitioned into smaller ones. A geometric figure is said to be symmetrical if there is at least one rigid motion (or symmetry operation) that can be performed on it that leaves the appearance (and apparent position) of the figure unchanged. (For example, in the plane, if a parallelogram is rotated 180° about its center, in its new position the figure appears exactly as it did in its original position. Only if a distinguishing mark such as a label on a vertex were added

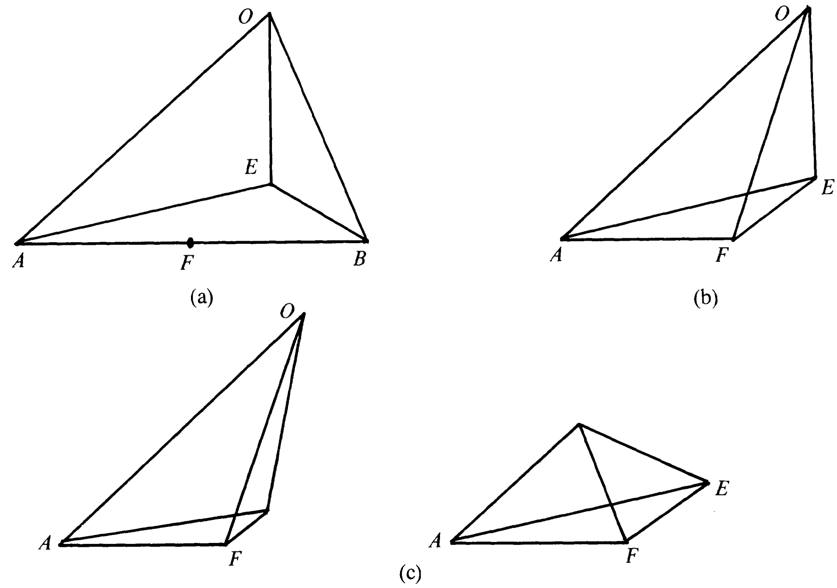
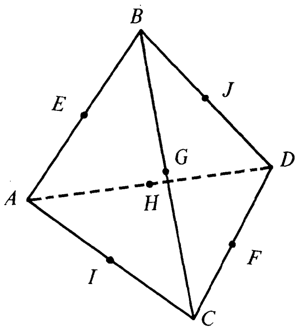


FIGURE 11. **(a) The tetrahedron of FIGURE 10, with F the midpoint of edge AB. If we take the edge length AB to be equal to 2, then $AE = BE = \sqrt{2}$, $OB = OA = \sqrt{3}$ and $OE = EF = AF = BF = 1$. The tetrahedron is symmetric about the plane through E, O, and F. (b) The plane through E, O, and F divides the tetrahedron into two mirror-image tetrahedra, $AEOF$ and $BEOF$ (we show only $AEOF$). Each has a two-fold rotation axis—the axis of the tetrahedron $AEOF$ passes through the midpoints of AO and EF. (c) The tetrahedron $AEOF$ can be partitioned into two congruent tetrahedra in two ways: by a plane through A, O, and the midpoint of EF, and by a plane through E, F, and the midpoint of AO.**

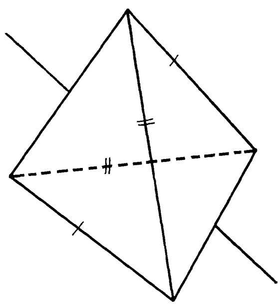
could you tell whether or not the parallelogram had been moved. Reflection in a diagonal of a parallelogram is not a symmetry operation unless the parallelogram is a rhombus.) A symmetrical object can be partitioned into congruent parts which the symmetry operation maps onto one another; if it has several symmetry operations, then there may be several ways to do this. The tetrahedron has the unusual property that these parts may be chosen to be tetrahedra. The relation between the symmetry of a tetrahedron and the ways in which it can be partitioned is shown in TABLE 1. Notice that if the operation is reflection in a plane or rotatory reflection (i.e., a rotation followed by a reflection as described in TABLE 1), then we obtain mirror-image pairs which, unless these new tetrahedra themselves have mirror symmetry, differ in the same way that right- and left-handed coordinate systems do. Since reflection cannot actually be performed in three-dimensional space, reflection is sometimes called an "improper" motion. Some authors distinguish between tetrahedra that fill space with "properly" congruent copies, and those, such as the tetrahedra of FIGURE 11(b), which must be accompanied by their mirror images.

The first systematic study of space-filling tetrahedra was carried out by D. M. Y. Sommerville (1879–1934), a geometer with deep interests in many fields of science. According to *The Dictionary of Scientific Biography*, "crystallography held a special appeal for him and crystal forms doubtless motivated his investigation of repetitive space-filling geometric patterns." The immediate inspiration for his study of tetrahedra was, however, an error made by a student. Sommerville wrote, "In the answer to the book-work question, set in a recent examination to investigate the volume of a pyramid, one candidate stated that the three tetrahedra into which a triangular prism can be divided are *congruent*, instead of only equal in volume. It was an interesting question to determine the conditions in order that the three tetrahedra should be congruent, and this led to the wider problem—to determine what tetrahedra can fill up space by repetitions" [**22**] (FIGURE 12). Sommerville wrote two papers on the subject. The first dealt with the wider problem; in the second he was concerned with the partition of triangular prisms into congruent tetrahedra, and whether these tetrahedra could fill all space.

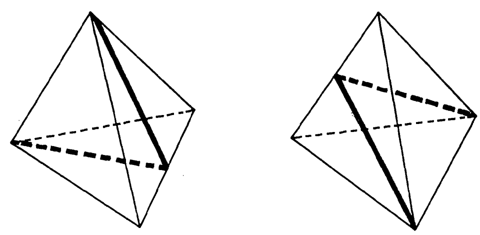
TABLE 1. **Tetrahedral symmetry and partitions of tetrahedra.**

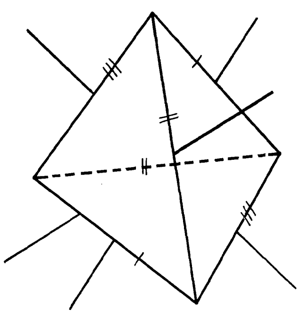


All tetrahedra described are represented schematically by the drawing at the left. Vertices of the tetrahedra are labeled as shown, with letters *A*, *B*, *C*, and *D*. Midpoints of the edges *AB*, *CD*, *BC*, *AD*, *AC*, and *BD*, are denoted *E*, *F*, *G*, *H*, *I*, and *J*, respectively. To show two edges are the same length, we mark them with the same symbol (either /, //, or ///), as is customary in elementary geometry.

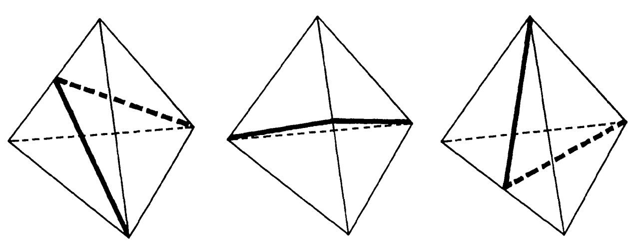


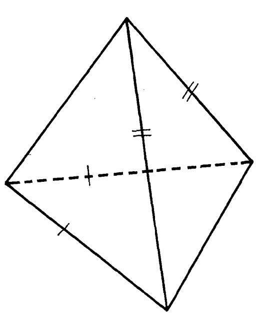
1. Symmetry 2. A single two-fold (180°) rotation axis through *E* and *F*. The tetrahedron can be partitioned into two asymmetric congruent tetrahedra in two ways, by plane *ABF* or plane *CDE*.





2. Symmetry 222. Three mutually perpendicular two-fold axes, through *E* and *F*, *H* and *G*, *I* and *J*. Each of these axes permits a partition of the tetrahedron into two congruent tetrahedra as described in 1. Since opposite edges are equal, each axis produces just one partition.





3. Symmetry *m*. A single mirror plane. This tetrahedron has two mirror-image scalene faces, *ABC* and *BAD*, and two isosceles faces, *BDC* and *ACD*. A plane through *A*, *B*, and *F* divides the tetrahedron into two asymmetric mirror-image tetrahedra.

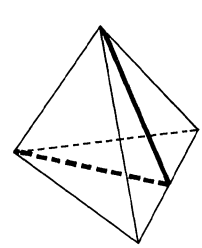
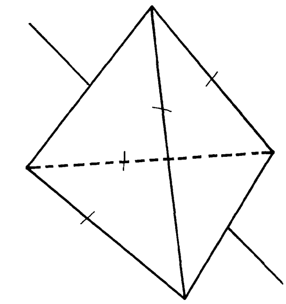
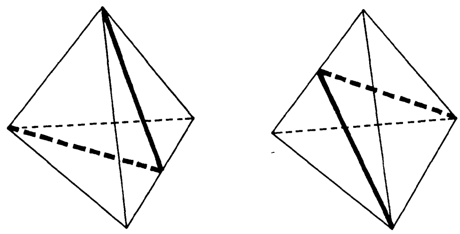
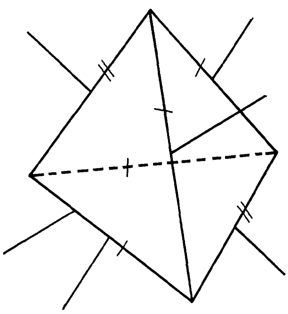


TABLE 1. **Tetrahedral symmetry and partitions of tetrahedra.**

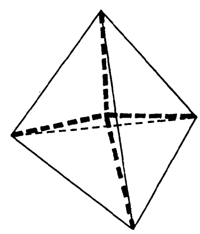


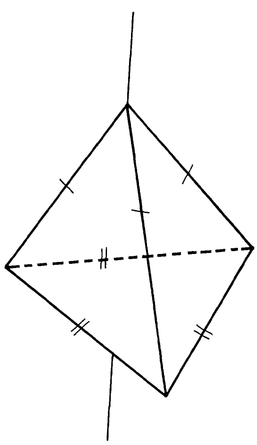
4. Symmetry 2*m*. This tetrahedron has **a two-fold axis *EF* which is the intersection of two perpendicular mirror planes** *ABF* and *ECD*. It can be partitioned in two ways into congruent tetrahedra with symmetry *m*.



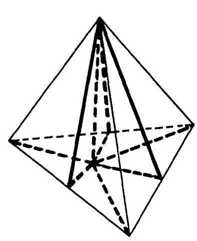


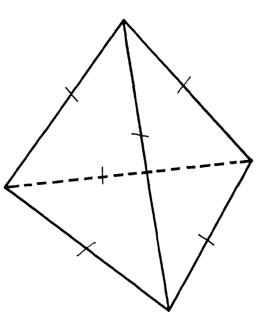
5. Symmetry $\bar{4}$*m*. This symmetry class contains classes 1–4: it has three two-fold axes and two mirror planes. It is generated by reflection in one of the mirror planes and by **a four-fold rotatory-reflection (denoted by $\bar{4}$): rotation 90° about *EF* followed (nonstop) by reflection in the plane through the center *O* of the tetrahedron perpendicular to *EF*.** This operation maps the four faces of the tetrahedron onto one another cyclicly. Thus if we join the vertices to *O* we obtain four congruent tetrahedra with symmetry *m*: *ABCO, BDCO, BADO,* and *ACDO.*





6. Symmetry 3*m*. This tetrahedron has **a three-fold axis** *BQ*, where *Q* is the center of face *ACD*, and **three mirror planes passing through it.** The tetrahedron can be partitioned into six asymmetric congruent tetrahedra.





7. The regular tetrahedron. All faces are equilateral. Each of *EF*, *HG*, and *IJ* is a $\bar{4}$ axis, and there are six mirror planes. The lines from each vertex to the center of the opposite face are three-fold (120°) rotation axes. The tetrahedron can be partitioned in all of the ways shown above—and in other ways as well (the discovery of which we leave for the reader).

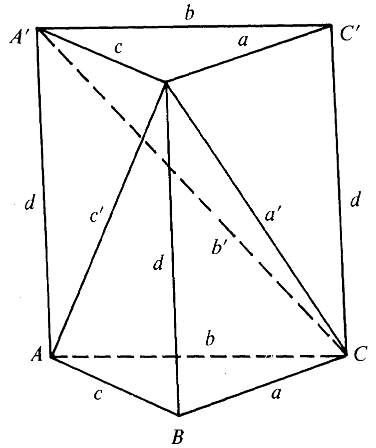


FIGURE 12. **A triangular prism $ABCA'B'C'$ can be partitioned into three tetrahedra, $ABCB'$, $B'A'CA$, and $A'B'C'C$.**

In the first paper, Sommerville defined a space-filling tetrahedron to be one which

(a) fills space with properly congruent copies such that
(b) the tetrahedra are juxtaposed face-to-face.

If a face of a tetrahedron is equilateral or isosceles, then it can be matched to the corresponding face of a properly congruent copy; otherwise it can be matched only if its mirror image also appears on the tetrahedron (consequently the tetrahedron itself must have mirror symmetry). With these definitions and observations, Sommerville classified space-filling tetrahedra into two kinds: (1) tetrahedra with mirror symmetry, and (2) tetrahedra without mirror symmetry all of whose faces are isosceles (FIGURE 13).

Sommerville then addressed the problem from both the global and local viewpoints. First, he applied Potaman's principle to the cube and discovered four tetrahedra of the first kind (FIGURE 14). Although, as we have seen, some of these tetrahedra can be partitioned further, Sommerville did not do so, evidently because the resulting tetrahedra would not satisfy condition (a). He then considered, in general, the ways in which tetrahedra of the first kind can be fitted together at a vertex. Using Bradwardinus's technique, he enumerated the triangular patterns that these tetrahedra would define on the surface of a sphere, and concluded that the four he had found the other way were the only ones possible. For some unexplained reason, however, he did not carry out a similar enumeration for tetrahedra of the second kind. Thus his claim to completeness is not justified.

(1) (2)

FIGURE 13. **According to Sommerville, there are two mutually exclusive requirements for a tetrahedron to fill space by the juxtaposition of properly congruent copies face-to-face: (1) the tetrahedron has mirror symmetry, or (2) the tetrahedron does not have mirror symmetry but all its faces are isosceles.**

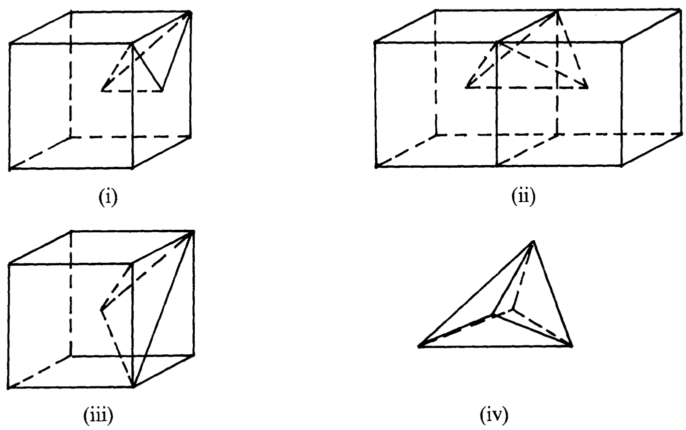


FIGURE 14. **The four Sommerville space-filling tetrahedra. (i) The first tetrahedron is that of** FIGURE 10. **(ii) The second is found by joining two vertices of a cube which share a common edge to the centers of two adjacent cubes, as shown. (If this tetrahedron is bisected along the cube face, we obtain two tetrahedra of the first type.) (iii) The third tetrahedron is obtained from the first by joining two tetrahedra along a common face through a cube vertex, face center and cube center. (iv) Since the second has $\bar{4}m$ symmetry, we can subdivide it into four congruent tetrahedra, each of which has mirror symmetry, as described in** TABLE 1.

In the second paper Sommerville showed that if the tetrahedra into which a triangular prism can be partitioned (FIGURE 12) are congruent, then $a = b' = c'$ and one of these additional relations among the edges must hold:

(i) $a' = b = c = d,\ 3a^2 = 4b^2$,

(ii) $a' = b = c$

(iii) $a' = b = d$ or equivalently $a' = c = d$,

(iv) $b = c = d$.

The four families of prisms defined by these relations are shown schematically in FIGURE 15.

He then asked whether these tetrahedra could fill all of space. The tetrahedron obtained from family (i) is the second of the four that he had found in the first paper, and he showed how the other three could be derived from it by partition. In each of families (iii) and (iv), one of the three tetrahedra is the mirror image of the other two, and so his definition of space-filling is not satisfied. Although the tetrahedra of family (ii) are properly congruent, Sommerville argued that the prisms of this family cannot fill space without their mirror images.

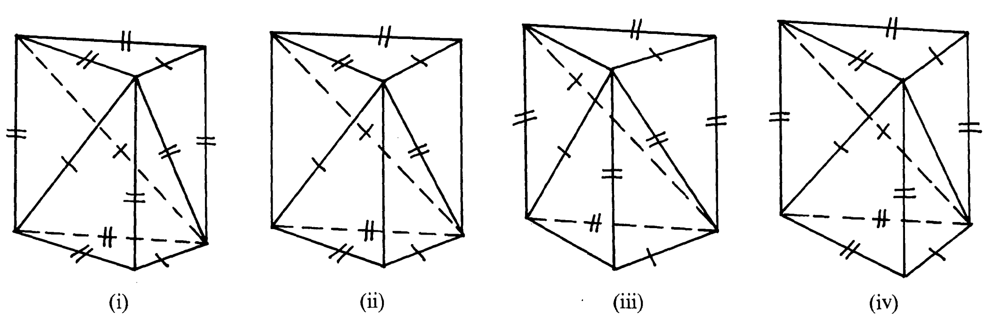
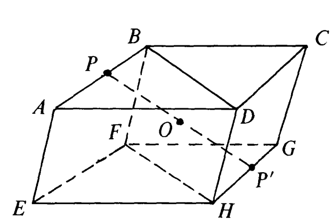


FIGURE 15. **Schematic drawings of Sommerville's four families of triangular prisms which can be partitioned into congruent tetrahedra. There is only one member of the first family (up to similarity); the other families are infinite.**

FIGURE 16.

A parallelepiped can be divided (in six different ways) into two mirror-image triangular prisms. The two prisms are related by inversion in the center O: if we join any point P of prism $ABDEFH$ to O and extend this line segment by length $|OP|$, we find the corresponding point P' of the prism $BCDFGH$.



Every triangular prism fills space. The easiest way to see this is to note that any parallelopiped can be partitioned into two mirror-image triangular prisms (FIGURE 16). Conversely, any triangular prism can be joined to its mirror image upside-down along a parallelogram face to form a parallelopiped. (This may have been Haüy's reason for choosing parallelopipeds, tetrahedra, and triangular prisms for his basic polyhedral units.) If we partition a parallelopiped into prisms and then tetrahedra, and stack the parallelopipeds face-to-face, their constituent parts are juxtaposed with their mirror images. A triangular prism can be juxtaposed face-to-face with *properly* congruent copies only if each parallelogram face has mirror symmetry or if one does and the other two are mirror images. The prisms of Sommerville's second family do not satisfy either condition and this may be why he did not consider them to be true space-fillers. (M. Goldberg has pointed out that prisms of this family can fill space with directly congruent copies of the tetrahedra in a helix-like pattern, as shown on the cover. Stacking copies of such a prism end-to-end we obtain prisms of infinite length, the cross-sections of which are equilateral triangles. Then the infinite prisms can be packed together as in FIGURE 3. But this space-filling is not face-to-face.)

Even so, Sommerville's argument seems a little curious. In general, the tetrahedra derived from the prisms of family (ii) do not satisfy either of his necessary conditions (1) or (2) for face-to-face space filling. But in the special case when $a = d$, the tetrahedron is of the second kind. When copies of these tetrahedra are assembled into triangular prisms, they cannot fill space without their mirror images, but this fact does not prove that there is no way they can do so. By calculating dihedral angles one can show that in fact they do not fill space face-to-face, but this raises again the question Sommerville left unresolved, the status of the tetrahedra of this kind which cannot be obtained from a triangular prism.

On the other hand it is remarkable that, as far as we are aware, all the known space-filling tetrahedra, regardless of the technique used to find them, can be obtained from Sommerville's four prism families. H. S. M. Coxeter discussed three tetrahedra which generate space-filling copies by reflection in their faces [**15**, p. 84]; these turn out to be Sommerville's first and second, and the partition of the first shown in FIGURE 11(b). H. L. Davies rediscovered the tetrahedra of Sommerville's fourth prism family and obtained a second family by partition [**16**]. He also showed how Sommerville's first and fourth tetrahedra can be derived from these by specializing edge and angular relationships, and found another by partitioning the latter. L. Baumgartner discovered Sommerville's first, second, and fourth tetrahedra and an additional one obtained from the second by bisecting it with a plane containing a two-fold rotation axis [**13**], [**14**]. M. Goldberg, restricting himself to properly congruent tetrahedra, partitioned Sommerville's second family in the two possible ways to obtain three families [**18**]; as we have already noted, these space-fillings are not face-to-face. The five tetrahedra found by E. Koch in her computer study of a class of crystallographically important Dirichlet regions [**20**] are the four Sommerville tetrahedra plus the tetrahedron found by Baumgartner. Whether there are any tetrahedral space-fillers that cannot be obtained by partitioning a triangular prism remains an open question.

More generally, we can ask whether every tetrahedral space-filler can be obtained by partitioning a parallelohedron. We can also ask, as we suggested at the beginning of this section, whether there exist tetrahedra which fill space in an irregular way. (Indeed, it is possible that such tetrahedra might even satisfy Sommerville's conditions (a) and (b).) More than 2300 years after Aristotle, the question of which tetrahedra fill space and which do not is still unresolved!

The general space-filling problem is still wide open. Challenging open problems abound; the answers will be important not only for mathematics but also for crystallography and other fields concerned with the partition of space. We do not know the shapes of space-fillers, except for the parallelohedra and certain other special classes; we do not even know the maximum number of faces a space-filler can have, though the number has been proved to be finite for one important general class. The largest number of faces known to occur in a convex space-filler is thirty-eight, as was recently found by the crystallographer P. Engel [**17**]. Even more generally, there is the problem of filling space with copies of two or more kinds of polyhedra. In Plato's words, "their combinations with themselves and with each other give rise to endless complexities, which anyone who is to give a likely account of reality must survey."

I would like to thank Branko Grünbaum, Susan Petrelli (Smith College, '82) and Lester Senechal for their helpful comments on the preliminary version of this paper, and Deedie Steele (Hampshire College, '81) for constructing excellent models of the Sommerville tetrahedra.

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Suggestions for further reading are provided for each section, along with appropriate comments.

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