Advanced Plane Topology From an Elementary Standpoint

A simple combinatorial approach to subtle topological premises of basic undergraduate analysis.

Donald E. Sanderson
Iowa State University
Ames, IA 50010

The title of Felix Klein's classic attempt to smooth the discontinuities between secondary and university mathematics [7] is paraphrased here in the hope of conveying the spirit of this paper and not for adding any guilt by association. Klein's objective was to show how university mathematics could enrich and elucidate secondary material. Turning half-circle from this point of view, our goal here is to show how undergraduate mathematics can reveal classical results of advanced plane topology which impinge on undergraduate mathematics but are seldom fully treated until graduate school, if ever.

At least three examples come to mind. First, most mathematics majors are exposed to Green's Theorem, which equates an integral on a simple closed curve to an integral over the region it bounds. The statement itself implicitly involves the Jordan Curve Theorem which guarantees that such a curve indeed bounds a unique finite region (see [2] page 243). Secondly, in elementary calculus the Intermediate Value Theorem is often used to show that $f(x)=0$ has a solution between a and b if $f(a)<0<f(b)$ and f is continuous. Elementary coordinate changes show that this is equivalent to existence of a solution of an equation $g(x)=x$ (a "fixed point" of g) where g maps an interval into itself. This is the one-dimensional Brouwer Fixed Point Theorem, a special instance of more general fixed point theorems that are useful for showing the existence of solutions to differential equations, economic models, etc. Finally, the implicit function theorem, necessary for justifying implicit differentiation, entails some form of Brouwer's Invariance of Domain Theorem (see footnote on page 281 of [4]). Students need not be burdened by complete proofs of all such topological preliminaries, but for those interested they should be more accessible.

Perhaps the simplest complete treatment of the classical results discussed here is in M. H. A. Newman's book [8]. The very last section of Hocking and Young [6] gives a modern version of J. W. Alexander's combinatorial approach [1] to some of these topics. However, each of these sources involve at least a modicum of algebraic topology. The approach taken here uses none, but could be used to motivate an introduction to algebraic topology. Little more is used than the barest facts about continuous functions and their behavior with regard to closed, compact (closed bounded) and connected sets. Most references are for the record (or the very curious), not for suggested reading, as many are relatively inaccessible. No apology is offered as that situation is the motivation for this paper!

The author wishes to acknowledge a considerable debt to Albrecht Dold for his excellent paper [5] which, somewhat in the spirit of Klein, was presented at a conference fostering interaction between universities and "Hochschulen" in Germany. Many of the basic ideas in what follows originated there and inspired this attempt to publicize and expand on his results.

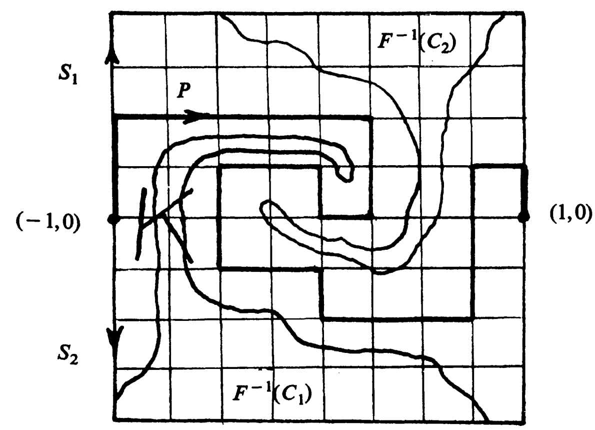
A Fundamental Result

A great deal of important topology of n-dimension Euclidean space E^n can be derived with surprising ease from what is sometimes called Alexander's Addition Theorem. A special case, sufficient for the topology of E^2, can be established by a simple parity (even-odd) argument similar to that used by Euler to solve the Königsberg bridge problem. For simplicity we establish some standard terminology and notation.

We call a continuous function a **map**. If $f: I \to X$ is a map of an interval I with ends a, b, then the image, $f(I)$, is called a **path** in X from $f(a)$ to $f(b)$. Observe that the continuous image of a path is again a path. A map f is called an **embedding** or **homeomorphism** onto its image if it is $1-1$ (i.e., its **inverse** f^{-1} exists) and f^{-1} is a map. Let R be the square region in E^2 with corners $(\pm 1, \pm 1)$. The region R has a square boundary $S = S_1 \cup S_2$ where S_1 is the upper and S_2 the lower half, each being a path joining $(-1,0)$ to $(1,0)$. We can now state and prove our key theorem which says, roughly, that if a path from x to y missing one closed set can be continuously deformed into one missing a second closed set without ever touching a point in both sets, then some path from x to y misses both sets.

ALEXANDER ADDITION THEOREM. *For $i = 1,2$, suppose C_i is closed in a space X, P_i is a path in $X - C_i$ from x to y and $F: R \to X - (C_1 \cap C_2)$ is a map with $F(S_i) = P_i$. Then some path $F(P)$ in $X - (C_1 \cup C_2)$ joins x to y.*

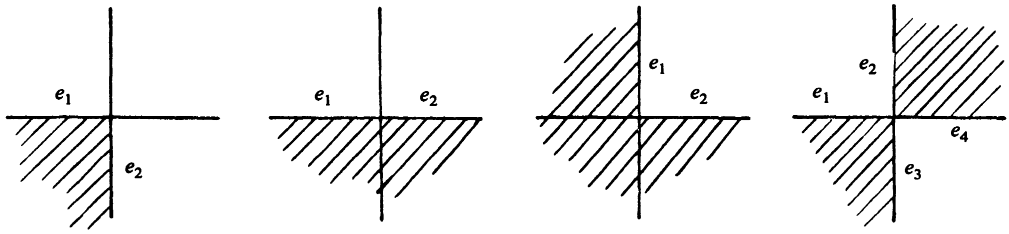
Proof. A picture of the square domain R of F appears in FIGURE 1; the map F is the continuous deformation referred to in the description of the theorem. Note first that $F^{-1}(C_2)$ has no points in common with $F^{-1}(C_1)$ or S_2. Since these sets are closed, R can be divided into



Pick a path P so $F(P)$ misses $C_1 \cup C_2$.

FIGURE 1.

finitely many squares by horizontal and vertical lines (including the axes) so no square of R meets both $F^{-1}(C_2)$ and $F^{-1}(C_1) \cup S_2$. Let K be the collection of squares meeting the latter and B the set of edges of the little squares lying on exactly one square of K. Then only an even number (two or four) of such edges have a common point (see FIGURE 2). All of S_2 is in B so $B - S_2$ has finitely many edges, with an even number meeting at each endpoint except for the ends, $(\pm 1, 0)$, of S_2. Starting at $(-1,0)$, we can trace out a path P along edges of $B - S_2$ which must terminate at $(1,0)$ since the even order condition assures that any other endpoint arrived at can be left by a different edge (if no edge is retraced, the path must indeed terminate). Each edge of P is an edge of a square of K meeting $F^{-1}(C_1) \cup S_2$ and not $F^{-1}(C_2)$, so P misses $F^{-1}(C_2)$. To show P also misses $F^{-1}(C_1)$, note that each edge e of P lies in $B - S_2$. Thus if



Edges e_i of B meet by twos or fours.

FIGURE 2.

$e \subset S = S_1 \cup S_2$, then $e \subset S_1$ and $F(e) \subset F(S_1) = P_1 \subset X - C_1$. If $e \not\subset S$, then e lies on two squares of R, both of which belong to K if e meets $F^{-1}(C_1)$, so then e could not be in B. Thus P misses $F^{-1}(C_1)$ and $F(P)$ is the desired path.

Before going on let us see what we have really accomplished. Suppose R is any collection of square regions which intersect, if at all, in a common edge or vertex (call such an R a **surface**). Let S_1 and S_2 be paths in R meeting only at their ends p, q (replacing $(\pm 1, 0)$ above). If each S_i is a union of odd order edges of R (edges lying on an odd number of squares of R) and no other such edge meets $F^{-1}(C_1 \cup C_2)$, then *the Alexander Addition Theorem still holds for surfaces*. Only minor changes are required in the proof: K is the collection of squares meeting $F^{-1}(C_1) \cup S_2$ as before and B is the set of odd order edges of K. Each vertex of B has even order. That is, if $e_1, \ldots, e_m$ are the edges of B meeting at a vertex v, then m is even. To see this, let $e_1', \ldots, e_n'$ be the remaining edges of K meeting at v. If we count the two edges of each square having v as a vertex, then the total count is even. But each e_i is counted $2n_i + 1$ times (being of odd order) and e_i' is counted $2n_i'$ times, so $\sum_{i=1}^{m}(2n_i + 1) + \sum_{i=1}^{n} 2n_i' = m + 2(\sum_{i=1}^{m} n_i + \sum_{i=1}^{n} n_i')$ is even and m must be even. The path P from p to q is constructed as before and the only other change needed is editorial (in the next to last sentence of the proof change "two" to "an even number of" and "both" to "all").

Extension to higher dimensions is analogous. Suppose, for example, that R is the cubical region in E^3 with corners $(\pm 1, \pm 1, \pm 1)$ and $S = S_1 \cup S_2$ its boundary surface where S_1 is the upper and S_2 the lower half. Instead of paths spanning the points x and y, the $P_i = F(S_i)$ are now images of the surfaces S_i each of whose boundaries is $S_1 \cap S_2$. We say S_i **spans** $S_1 \cap S_2$. Slice up R parallel to coordinate planes and define K and B as before (except K is a collection of cubes, B is a set of odd order faces of K and we now seek a surface P **spanning** the square $S_1 \cap S_2$ in the sense that $S_1 \cap S_2$ is the union of the odd order edges of P). Counting as in our discussion of surfaces, we find that every edge of $B - S_2$ except for those in $S_1 \cap S_2$ has even order in $B - S_2$. Remove (or blacken) a face of $B - S_2$ having an odd order edge and continue doing so as long as what is left (unblackened) of $B - S_2$ has a face with an odd order edge (i.e., an edge lying on an odd number of unblackened faces of $B - S_2$). When there are none left, the black faces form a surface $P \subset B - S_2$ whose odd order edges (those on an odd number of black faces) form $S_1 \cap S_2$ and whose image $F(P)$ misses $C_1 \cup C_2$.

Three Important Applications

We now apply the Alexander Addition Theorem to get three famous results. In the first case choose for the space X of the theorem the set S (the boundary of the square region R of FIGURE 1), then choose C_1 and C_2 to be the bottom and top edges of R respectively and take $P_i = S_i$. If there is a map $F: R \to X - (C_1 \cap C_2) = X$, which is the identity when restricted to $X \subset R$, then the Alexander Addition Theorem gives a path $F(P)$ joining $(\pm 1, 0)$ in $X - (C_1 \cup C_2)$, which is impossible. Thus no such extension of the identity (called a **retraction** of R to $X = S$) exists. We state this as a theorem.

NO RETRACTION THEOREM. *No map of R to S maps each point of S to itself.*

In short, S is not a **retract** of R. It is in fact an easy exercise to show that the same is true for any disk (space homeomorphic to R). For example, the unit circle $C=\{(x,y)|x^2+y^2=1\}$ is not a retract of the unit disk $D=\{(x,y)|x^2+y^2\leqslant 1\}$. We also leave as an exercise the proof of the No Retraction Theorem for surfaces spanning S and for higher dimensions.

Another variant of the Intermediate Value Theorem of calculus says that a map of an interval into the reals which maps the endpoints onto themselves takes on every value in the interval. A two-dimensional version of this follows from the No Retraction Theorem. Also, as in one-dimension, Brouwer's Fixed Point Theorem in two dimensions (every map of a disk into itself has a fixed point) is an easy consequence.

INTERMEDIATE VALUE THEOREM. *If f maps the unit disk D into E^2 and is the identity on the unit circle C, then $D\subset f(D)$, that is, f takes on every "value" in D.*

Proof. If $p\in D-f(D)$, then f followed by projection from p to C retracts D to C. But, as we just observed, C is not a retract of D.

BROUWER FIXED POINT THEOREM. *If f maps the unit disk D into E^2, then either $f(p)=p$ for some point p in D or $f(p)=\lambda p$ for some p on the unit circle C and $\lambda>1$. Thus if $f(D)\subset D$ or even $f(C)\subset D$, then f has a fixed point.*

Proof. Define, as usual, $\|v\|=\sqrt{x_1^2+x_2^2}$ if $v=(x_1,x_2)$ in E^2; then set $g(v)=2v-f(2v)$ if $\|v\|\leqslant 1/2$ and $g(v)=(v/\|v\|)-2(1-\|v\|)f(v/\|v\|)$ if $1/2\leqslant\|v\|\leqslant 1$. Then $g:D\to E^2$ is a map which is the identity on C and by the Intermediate Value Theorem $g(v)=(0,0)$ for some v in $D-C$. If $\|v\|\leqslant 1/2, f(2v)=2v$ and $p=2v$ is the required point of D. If $\|v\|>1/2, f(v/\|v\|)=(2-2\|v\|)^{-1}v/\|v\|=\lambda v/\|v\|$ and $p=v/\|v\|$ is the required point of C since $\|v\|>1/2$ implies $\lambda=(2-2\|v\|)^{-1}>1$.

Connectivity and Components

At this point we have successfully investigated the fixed point theorem which is implicit in the intermediate value theorem. The topological ideas implicit in Green's Theorem and the implicit function theorem require some further development. We will proceed next to a study of the Jordan Curve Theorem which is buried in Green's Theorem. The Jordan Curve Theorem, which we shall state precisely and prove later, says that a simple closed curve divides the plane into exactly two pieces, one inside and one outside. The appropriate topological ideas with which to study "pieces" are the concepts of connectedness and components. Our purpose in this section is to present and develop these ideas.

We begin by noting that the nonexistence of the function F in the Alexander Addition Theorem just yielded three important results with marvelous ease. Our study of connectedness and components will show what can be done when such maps F do exist. It turns out that the hypotheses of the Alexander Addition Theorem are commonly satisfied. Each P_i is a continuous image of an interval $[a,b]$ which is homeomorphic to S_i so a map of $S=S_1\cup S_2$ taking S_i to P_i always exists. However, extending this to a map F of R into X is the problem. Existence of such extensions is important for many other reasons in topology and analysis, so we codify this with a definition. If every map of S into a space extends to a map of R, we call the space **simply connected**. Thus the Alexander Addition Theorem applies whenever $X-C_1\cap C_2$ is simply connected. (Even when it is not, our comments on surfaces may apply.) In particular the Alexander Addition Theorem applies if $X-C_1\cap C_2$ is "starlike", i.e., homeomorphic to a set $Y\subset E^n$ having a point p joinable to every point $y\in Y$ by a straight line segment $py\subset Y$. For if $F:S\to Y$ is a map, it extends to a map of R into Y which takes the origin O to p and each segment Oq, $q\in S$, linearly onto the segment $pF(q)$. Note that by the No Retraction Theorem, S itself is not simply connected, since an extension of the identity would be a retraction.

Turning our attention to the complementary open sets $U_i=X-C_i$, we can discover a simple and useful method for finding and counting the "connected parts" of $U_1\cap U_2$. Euclidean spaces have arbitrarily small neighborhoods whose points are joined by paths within the neighborhood. For such **locally path connected** spaces, an open set U is **connected** iff it is **path connected**, i.e., iff any two points of U are ends of a path in U. A **component** ("connected part") of an open set V is the set of all points of V joinable to a given point by a path in V.

COMPONENTS OF INTERSECTIONS THEOREM. *Let C_1 and C_2 be closed in a space X and let $U_i = X - C_i$. If $U_1 \cup U_2$ is both simply and locally path connected, then each component of $U_1 \cap U_2$ is the intersection of a component of U_1 and one of U_2.*

Proof. If $x \in K$, a component of $U_1 \cap U_2$, and K_i is the component of U_i containing x, then $K \subset K_1 \cap K_2$ since a path joining y to x in $U_1 \cap U_2$ does so in U_1 and U_2 as well. Conversely, if $y \in K_1 \cap K_2$, then $y \in K_i$ so $U_i = X - C_i$ contains a path P_i joining the points x, y of $U_1 \cap U_2 = X - (C_1 \cup C_2)$. By the Alexander Addition Theorem there is a path $F(P)$ in $X - (C_1 \cup C_2) = U_1 \cap U_2$ between x and y. But then $y \in K$ and $K_1 \cap K_2 \subset K$ which means $K_1 \cap K_2 = K$. This completes the proof.

Let $k(U)$ denote the number of components of U. The result just proved implies, as we show next, the very useful formula:

$$k(U_1) + k(U_2) = k(U_1 \cup U_2) + k(U_1 \cap U_2). \tag{1}$$

The observant student of mathematics will notice this same formula in many other contexts, e.g., where $k(U)$ denotes cardinality of finite sets, dimension of linear subspaces, measure, Euler characteristic (Vertices − Edges + Faces), etc.

COUNTING COMPONENTS THEOREM. *The formula $k(U_1) + k(U_2) = k(U_1 \cup U_2) + k(U_1 \cap U_2)$ holds if the U_i are open and $U_1 \cup U_2$ is both simply and locally path connected.*

Proof. Let $W_1, \ldots, W_n$ be a list of all components of U_1 and U_2 and p_m the number of intersecting pairs from each initial list $W_1, \ldots, W_m$. Then $k(U_1) + k(U_2) = n$ and, by the Components of Intersections Theorem, $k(U_1 \cap U_2) = p_n$ since each nonempty $W_i \cap W_j$ is a component of $U_1 \cap U_2$ as well as of any subset of $U_1 \cap U_2$ containing $W_i \cap W_j$. Thus our formula can be written $k(\cup_{i=1}^n W_i) = n - p_n$. To prove it, we use induction on n. Since $k(W_1) = 1 = 1 - p_1$ is trivial, we assume $k(\cup_{i=1}^m W_i) = m - p_m$ for $1 \leqslant m < n$. By definition of p_m, W_{m+1} meets $p_{m+1} - p_m$ of the components $W_1, \ldots, W_m$. If no component K of $\cup_{i=1}^m W_i$ contains two (or more) of these $p_{m+1} - p_m$ W_i, then W_{m+1} meets exactly $p_{m+1} - p_m$ such components K and unites them into one component of $\cup_{i=1}^{m+1} W_i$ so that $k(\cup_{i=1}^{m+1} W_i) = (m - p_m) - (p_{m+1} - p_m) + 1 = (m+1) - p_{m+1}$, completing the proof. But a component K of $\cup_{i=1}^m W_i$ must lie in a component K' of $U_1' = \cup_{i \neq m+1} W_i$. The Components of Intersections Theorem applies to U_1' and $U_2' = W_{m+1}$ since $U_1' \cup U_2' = U_1 \cup U_2$ and the U_i' are open because the components W_i of open locally path connected sets are open. So, unless it is empty, $K' \cap U_2'$ is a component of $U_1' \cap U_2'$. Thus $W_{m+1} = U_2'$ meets only one W_i in K', hence only one in $K \subset K'$ (at most). So our formula is established.

The **n-sphere** S^n is the set of points in E^{n+1} a unit distance from the origin: S^0 is a two-point set, S^1 a circle, S^2 an ordinary sphere, etc. Path connectedness is often called **0-connectedness** since it requires that each map of S^0 extend to a map of the interval $[-1, 1]$ spanning S^0. Somewhat more general than simple connectedness is the property of **1-connectedness** which requires that each map of a square boundary, or equivalently of S^1, extend to a surface spanning S^1. Extension to 2-connected and beyond is more complex since there are boundary surfaces other than the 2-sphere (e.g., torus, double torus, etc.). There is an interesting and important relation between this "dimensional" connectedness of a suitable set and of its complement in E^n or S^n called Alexander's Duality Theorem. Our final application of the Alexander Addition Theorem is the proof of this relation for an arc in S^2. An arc is 0- and 1-connected (it is starlike) and the Alexander Duality Theorem amounts to saying the same is true for its complement. This result implies that the addition formula (1) is valid for open sets $U_i \subset S^2$ if the complement of

$U_1 \cup U_2$ is an arc. This can be seen by using our remarks about surfaces in place of the Alexander Addition Theorem to extend our two results about components. The same is true with E^2 in place of S^2 since adding a point "at infinity" to E^2 gives S^2 and removing a point from S^2 does not affect connectedness of open sets.

ALEXANDER DUALITY THEOREM. *The complement of an arc in the plane or 2-sphere is path connected.*

Proof. From the comment preceding the theorem it suffices to prove it for S^2, so let $A = h([-1,1])$ be an arc embedded in S^2 by a homeomorphism h. Two points x, y of $S^2 - A$ are joined by a path P in $S^2 - h(s)$ for any s in $[-1,1]$ since $S^2 - h(s)$ is homeomorphic to E^2. In fact, since $h^{-1}(P)$ is a closed set in $[-1,1]$ not containing s, P misses $h([r,t])$ for some $r \leqslant s \leqslant t$ with $r < s$ unless $s = -1$, and $s < t$ unless $s = 1$. Thus if we choose s to be the least upper bound of all numbers n in $[-1,1]$ such that $h([-1,n])$ misses a path from x to y, then $s > -1$ and some path P_1 from x to y misses $C_1 = h([r,t])$ where $r < s \leqslant t$. By our choice of s there is a path P_2 from x to y missing $C_2 = h([-1,r])$. If $X = S^2$, then $X - (C_1 \cap C_2) = S^2 - h(r)$ and $F: R \to X - (C_1 \cap C_2)$ with $F(S_i) = P_i$ for $i = 1, 2$ exists by the construction at the end of the paragraph following the proof of Brouwer's Fixed Point Theorem ($S^2 - h(r)$, being homeomorphic to E^2, is starlike). Thus the Alexander Addition Theorem gives a path from x to y in $X - (C_1 \cup C_2) = S^2 - h([-1,t])$. But if $s < 1$, then $s < t$ contrary to the choice of s, so $s = 1 = t$ and $S^2 - h([-1,t]) = S^2 - A$ is path (or 0-) connected.

The same pattern of proof, using the Alexander Addition Theorem in higher dimensions, shows that the complement of an arc A in S^2 is also 1-connected. Instead of joining given points x, y in $S^2 - A$ by paths, we extend a given map of the square $S_1 \cap S_2$ into $S^2 - A$ to a surface spanning $S_1 \cap S_2$. The P and P_i in the preceding proof are now the images of these surfaces. All extensions needed to complete the proof in the same pattern exist by the construction following Brouwer's Fixed Point Theorem, since, as noted before, S^2 minus a point is starlike.

Indeed, the same type argument proves duality for a disk $D = H(R)$ in S^2 (or E^2) where H is a homeomorphism and R is the square region in E^2 with corners $(\pm 1, \pm 1)$. For example, to show $S^2 - D$ is connected, observe that for $-1 \leqslant s \leqslant 1$, instead of a point $h(s)$ in the above proof, we have an arc $h(s) = H(s \times [-1,1])$. But duality for an arc in S^2 guarantees the extensions needed to complete the proof as before using the Alexander Addition Theorem for surfaces.

The Jordan Curve Theorem

Mathematicians and others long assumed that a simple closed curve (the embedding of a circle) in the plane separates the plane, as does a circle, into two connected pieces and is the boundary of each. In 1865 the German mathematician, Carl Neumann, in a book on integration asked for an explicit proof of this. Over twenty years later in 1887 a French mathematician, Camille Jordan, published a "proof" which was not valid even for a simple closed polygon! We commemorate this pioneering but shaky mathematics by continuing to call what he attempted to prove the Jordan Curve Theorem. It was almost another twenty years before the American topologist Oswald Veblen gave a complete valid proof in 1905. The Dutch mathematician L. E. J. Brouwer extended it to n-dimensional space in 1912 and in 1916 Alexander announced his Duality Theorem which extended it further.

The solution of the original plane problem is made simple by the Alexander Duality Theorem and the formula (1). It amounts to applying this formula to cases where $U_1 = E^2 - C$ and U_2 is any connected open set known to be separated into two components by the curve C. (Dold's proof, mentioned in the introduction, is based on the assumption that the entire curve C lies in such a set U_2. Thus Dold shows a global bisection of space assuming a global bisection of a neighborhood of the curve whereas we obtain a global bisection of space by using a known local bisection by certain auxiliary curves.)

Jordan Curve Theorem. *If C is a simple closed curve in the plane E^2, then $E^2 - C$ has two components and C is the boundary of each.*

Proof. If K is a component of $E^2 - C$ and U an open set containing a point x of C, then $C - U$ is contained in a subarc A of C. By the Alexander Duality Theorem, $E^2 - A$ is connected and so contains a path $P = f([-1,1])$ from $x = f(-1)$ to a point $y = f(1)$ of K. The closed set $f^{-1}(C)$ has a maximum m, $-1 \leqslant m < 1$ and $f(m) \in U$ so by continuity some interval about m maps into U. If n is a point of this interval and $n > m$, then $f(n) \in U$ and $f([n,1])$ is a path in $E^2 - C$ joining y to $f(n)$. Thus $f(n) \in K$ and x is in the closure, $\bar{K}$, of K. Since components of $E^2 - C$ are open, $C = \bar{K} - K$ is the boundary of K.

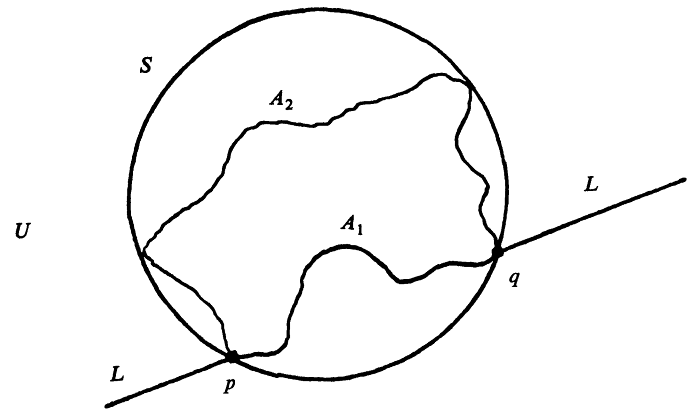


Figure 3.

To prove separation, let S be a circle containing at least two points p, q of C and whose exterior U contains no point of C (see Figure 3). One can imagine how S may be found by thinking of shrinking a circle inside which the bounded set C lies until it first touches C at a point p and further shrinking it with p fixed until it touches a second point q. The existence of p and q is assured since C is closed. Let A_i $(i = 1,2)$ be the arcs of C with ends p and q, L the part of the line through p and q not lying between them and $L_i = L \cup A_i$. Three calculations of component numbers complete the proof:

(i) If $U_1 = U$ and $U_2 = E^2 - L_i$, then $U_1 \cup U_2 = E^2 - A_i$ and, as observed prior to the Alexander Duality Theorem, our component formula (1) applies giving $k(E^2 - L_i) = k(E^2 - A_i) + k(U - L) - k(U) = 1 + 2 - 1 = 2$.

(ii) If $U_i = E^2 - L_i$, then $U_1 \cup U_2 = E^2 - L$ is simply and locally path connected since it is open and starlike (in fact it is homeomorphic to E^2). Thus by the Counting Components Theorem, $k(E^2 - L_1 - L_2) = k(U_1 \cap U_2) = k(E^2 - L_1) + k(E^2 - L_2) - k(E^2 - L) = 2 + 2 - 1 = 3$.

(iii) If $U_1 = E^2 - C$ and $U_2 = E^2 - L_1$, then $U_1 \cup U_2 = E^2 - A_1$. As in (i), the component formula (1) applies and gives $k(E^2 - C) = k(E^2 - A_1) + k(E^2 - L_1 - L_2) - k(E^2 - L_1) = 1 + 3 - 2 = 2$ as required.

Proving the Jordan Curve Theorem seems to require knowing that an arc does not separate the plane. Thus, although there are other proofs at least as elementary (see pp. 100–104 of **[10]**), some form of the Alexander Duality Theorem appears to be unavoidable and it was this that caused the most complication in the proof given here. Most of the complication and clutter is caused by the necessity of extensions to surfaces and higher dimensions which at the same time suggest the methods may generalize to any finite dimension. In fact this is so, and an oft-quoted principle of George A. Polya is at work here: a more general problem may have a simpler solution. The same methods, devoid of the necessity for extensions to surfaces and higher dimensions, are used in **[9]** to prove all these results in n-dimensional space. However, the latter is less intuitive, of course, since it is difficult to visualize beyond three dimensions.

Many applications of the Jordan Curve Theorem require only the polygonal case for which an easy intuitive proof can be found in [3]. An even simpler and rigorous proof (using induction on the number of edges) can be given if the edges are all horizontal or vertical. Try it!

The Invariance of Dimension

There are other important and easy applications of the Alexander Addition Theorem. For example, a half-dozen properties known as Phragmen-Brouwer Properties (see p. 359 of [6]) are easily verified for spaces which are both simply and locally path connected. We close by showing how the Jordan Curve Theorem and the Alexander Duality Theorem imply invariance of domain and dimension.

The inverse image of an open set with respect to any map is open (in fact this is often taken as the definition of continuity) and the image of an open set under a homeomorphism is open (since its inverse is continuous) but not necessarily under an embedding (homeomorphism with a subset). For example, no open subset of the reals maps onto an open set in E^2 under the natural embedding of the reals onto the x_1-axis. However, Euclidean spaces are somewhat unusual in that any embedding of an open subset of E^n (or S^n) in E^n (or S^n) is open. This is known as the Invariance of Domain of Euclidean spaces. Besides useful consequences in analysis, it has the reassuring topological consequence, called Invariance of Dimension, that E^m is not homeomorphic to E^n if $m \neq n$. This fact, first proved by Brouwer in 1911, was especially reassuring then because in 1890 Giuseppe Peano had destroyed the then current concept of dimension by showing that a 2-dimensional square disk is the continuous image of a 1-dimensional interval (in fact, any n-dimensional disk is) and previously Georg Cantor had shown that the points of a line can be put in 1-1 correspondence with those of a plane! The 2-dimensional version of this invariance is easily proved using the Jordan Curve Theorem and the Alexander Duality Theorem for disks.

INVARIANCE OF DOMAIN THEOREM. *If U is open in E^2 and h embeds U in E^2, then $h(U)$ is open.*

Proof. If p is a point of U, then U contains a circle C about p along with its interior I. It suffices to show that $h(I)$ is open. By the Jordan Curve Theorem, $E^2 - h(C)$ has two (open) components. Let V be the one containing $h(p)$ and W the other. Then the connected set $h(I)$ lies in V, and W contains no point of the disk $h(C \cup I)$. The complement of this disk contains W, is contained in $V \cup W$, is connected by the Alexander Duality Theorem for disks, and so must equal W. Then $V = h(I)$ and $h(I)$ is open.

INVARIANCE OF DIMENSION THEOREM. *The real line E^1 is not homeomorphic to the plane E^2.*

Proof. If $g: E^2 \to E^1$ is a homeomorphism and $f: E^1 \to E^2$ is the natural embedding of E^1 onto the x_1-axis of E^2, then $f \circ g = h$ embeds the open set $U = E^2$ and $h(U)$ is not open, contrary to the Invariance of Domain Theorem.

Thus a 1-1 map of E^1 onto E^2 must have a discontinuous inverse. In fact such a map cannot exist but a slightly more sophisticated argument is required. In summary, Cantor exhibited a 1-1 function from E^1 onto E^2. Peano's example yields a continuous function from E^1 onto E^2, but no such function can be both 1-1 and continuous.

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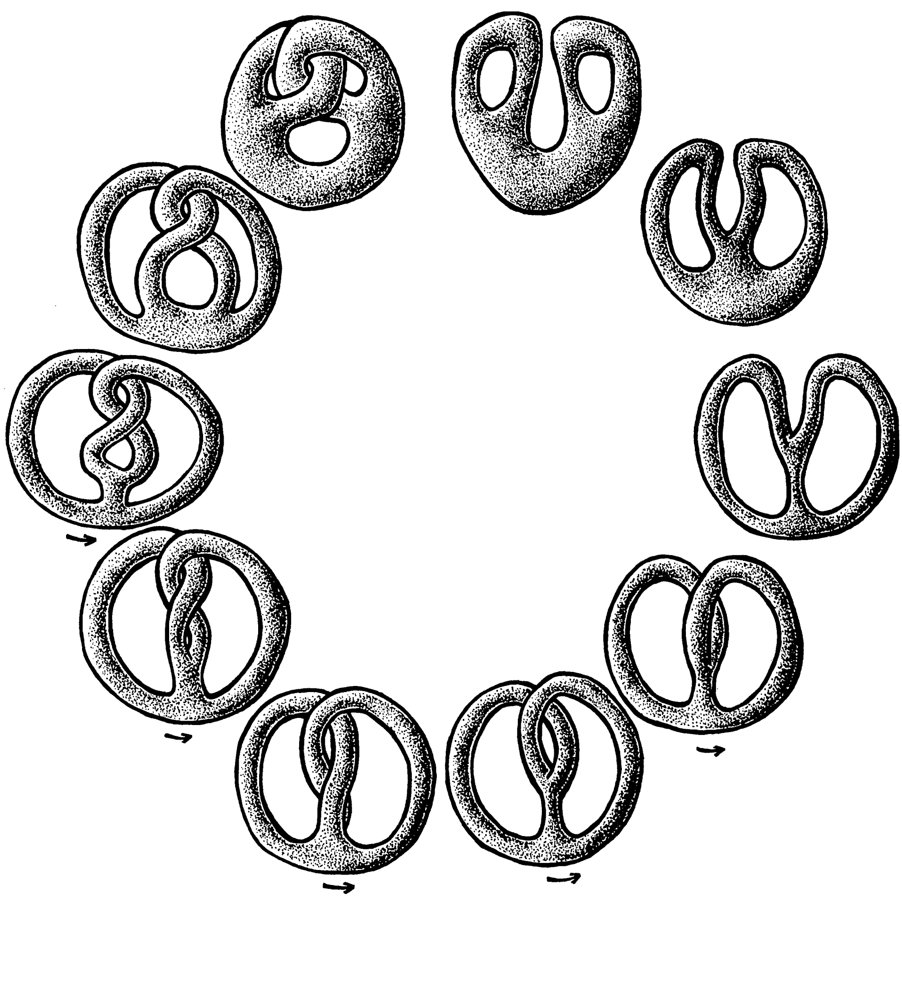

Dan Kalman
University of Wisconsin
Green Bay, WI 54302