

The King Chicken Theorems

A flock of results about pecking orders, describing possible patterns of dominance.

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Fact: In any barnyard of chickens, in every pair of chickens, one is dominant over the other. The dominant chicken asserts its dominance by pecking the other chicken on the head and neck, whence the phrase “pecking order.” However, it rarely happens that this pecking order is linear. That is, it is rare that there is a first chicken who pecks all the others, a second chicken who pecks all but the first, and so on. Question: Given that the pecking order may not be linear, is there still a reasonable way to designate a most dominant chicken, i.e., a king? In this paper, we analyze at length one promising king definition, and look briefly at several others.

This material illustrates nicely the delights and pitfalls of applied mathematics. One delight is that a model set up to describe one situation may well describe several others as well. A pitfall is that the mathematics done with the model, though good mathematics, may not really be of much use in the situation being modeled. It is not merely a matter of irrelevant theorems. It may be a matter of the wrong mathematical assumptions or even the wrong definitions. This last is a very subtle matter because it may be far from obvious that the definitions are wrong. Specifically, we wish to define a king chicken in such a way that the definition implies a strong sense of dominance and also implies that every flock of chickens has a king. We give such a definition and such a theorem—a definition and a theorem due to the mathematical sociologist H. G. Landau in 1953 [5]. But a king should also be unique, or at least rare. It is necessary to ask if our definition assures this. We are led to discover and prove many theorems which say that kings are all too common. In short, the theorems have negative implications about the usefulness of the definition. But we would not even have discovered this if we had not done some mathematics with the definition, and done it with this sort of testing in mind.

This lesson, that a model needs to be judged by the theorems it keeps, is perhaps the main point of this paper. I must confess, however, that for me the theorems themselves are also a main point; the pure mathematician in me finds them interesting in their own right. Also, their proofs illustrate some interesting techniques, for instance, duality and induction from n to $n+2$. In any event, good modeling does not stop with wrong definitions, no matter how pretty the theorems they give. One must look for better definitions and/or better mathematical assumptions, and then start proving theorems again. We say a bit about this second round towards the end of the paper.

This paper is written primarily for undergraduates, both in the style of proofs and in the amount of interpretation of theorems. There are problems stated later which could lead to good undergraduate projects. Indeed, the paper developed from work begun with undergraduates—my 1977 Graph Theory class at Princeton. Some of the results appear to be new.

A word about sex: there are female chickens, called hens, and there are male chickens, called roosters. To this natural division mankind has also added neuter chickens, called capons. These divisions are important to chickens. Chickens of the same sex residing together always form a pecking order, but roosters and hens rarely peck each other. Thus the “Fact” stated at the beginning of this article is not quite correct; one must restrict one’s attention to a single sex. Since a traditional barnyard has several hens but only one or two roosters, pecking is usually associated with hens, both in common parlance (“henpecked”) and in scholarly articles. For this reason several people have suggested rather heatedly that I should revise this article by replacing “king” throughout with “queen.” However, since all chickens peck, I have stuck with king. I use this as a general term for leader. I have also avoided all use of either male or female pronouns for chickens.

For a delightful and informative essay on chicken society in general and pecking in particular, one should read the original article on the subject by Schjelderup-Ebbe [16].

The Model

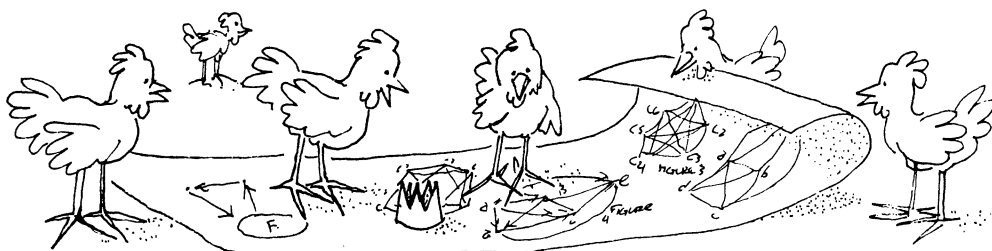
By a **flock** F of chickens we mean a nonempty (finite) set of chickens which has the property that for any two chickens in it, exactly one pecks the other. Formally, we should write $F=(C,P)$, that is, a flock is not just the set C of chickens, but a binary pecking relation P as well, where $(c,d) \in P$ iff c pecks d . However, it is more natural, and should not be confusing, if we write “chicken c is in F ,” rather than “chicken c is in the chicken set C of flock F .” Likewise, we often write $c \in F$ rather than $c \in C$.

We will sometimes want to refer to P alone, so let us call it the **pecking order**. Also, let us call any individual ordered pair in P a **pecking pair**. Sociologists use the term “peck right,” but in what I have read it seems to mean pecking order or pecking pair or both, depending on context.

In general, chickens will be denoted by c, d, a, b or sometimes c_1, c_2, \dots, c_n . Flocks will be F and G . Usually there will be exactly one flock under consideration, F , and all chickens mentioned will be understood to belong to F .

It is helpful to think of a flock as a **directed graph**. A directed graph consists of a set of vertices and a set of directed edges between various pairs of vertices. In our case the chickens become the vertices, and there is a directed edge from vertex c to vertex d if and only if chicken c pecks chicken d . Clearly, any binary relation can be represented by a directed graph in this way. The graph of a flock has an additional property: between every pair of vertices there is exactly one directed edge. Such a directed graph is often called **complete**. Flocks of chickens are not the only objects modeled by such graphs. Suffice it to say for now that the usual name in the literature for such a complete directed graph is “tournament.” The theorems we will prove are good for all complete directed graphs, and hence for all the situations they model. Perhaps then we should use standard graph-theoretic terms like “graph” and “vertex.” However, I have decided to stick to chicken-theoretic terms like “flock” and “chicken.” It helps keep firmly in mind the origins of our study, and it’s more fun.

Now, how shall we define king? Surely, if chicken c pecks all other chickens in its flock, it should be called a king, but such a c does not always exist. For instance, suppose there are n



chickens and they peck each other cyclically; that is c_1 pecks c_2, \dots, c_{n-1} pecks c_n , and c_n pecks c_1 . Then no matter what the other pecking pairs are, every chicken is pecked by at least one other. So this first suggestion for a definition is too narrow. Yet, if a chicken is to be called a king, surely it should dominate every other chicken, at least indirectly, and not too indirectly. Landau's idea was to allow domination in two steps as well as directly in one. Landau did not introduce a term for his concept, but we will.

DEFINITION. Chicken c in flock F is a **king** if for every other chicken d in F , either c pecks d , or there is some third chicken b such that c pecks b and b pecks d .

Since we will eventually find that this king concept is not a good one, it is important to say a bit more about what Landau did with it. His main concern [3, 4] was to measure how hierarchical a pecking order will be, based on various assumptions about how the individual pecking pairs are determined. He introduced the idea of domination in one or two steps, almost as an afterthought, as the last topic in a third paper [5] dealing with some related but more purely mathematical ideas. By introducing this 2-step concept and proving Theorem 1 below, Landau surely intimated that the concept might be useful for studying levels of dominance. However, he never explicitly discussed the problem of designating a most dominant chicken. Moreover, he did remark that his 2-step concept need not pick out a unique chicken. Thus Landau, who is dead, might well have objected to our use of the work "king" for his concept. So the negative results below about kings can in no way be regarded as a criticism of Landau's work.

Existence Theorems

The miraculous thing about Landau's definition is that there is always such a king, and that the proof is easy. First, a few further conventions: For $c \in F$, let S_c be the set of chickens c pecks (Submissive to c) and let D_c be the set which peck c (Dominates c). Clearly, S_c , D_c , and $\{c\}$ partition F . The partition is illustrated in FIGURE 1. In this figure and henceforth, a "balloon"

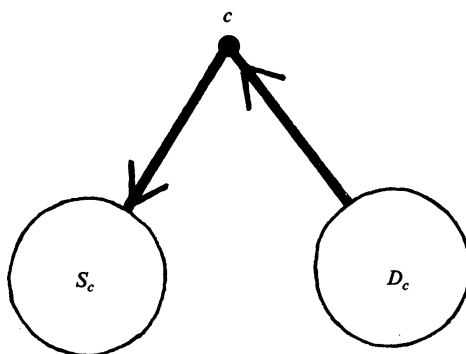


FIGURE 1.

represents a set of vertices, and a thick arrow between a single vertex and a balloon indicates that all edges between the single vertex and those in the balloon are directed like the thick arrow.

THEOREM 1 (Landau [5], p. 148). *Every flock of chickens has a king.*

Proof. Let $s(c)$ be the number of chickens c pecks. Let c be a chicken for which $s(c)$ is maximum. We will prove that c is a king. Suppose not. Then some chicken $d \in D_c$ is not dominated in two steps by c . That is, no chicken in S_c pecks d . Thus d pecks all chickens in S_c . But d also pecks c . So $s(d) > s(c)$, a contradiction.

COROLLARY 2. *Let s be the maximum number of chickens pecked by any chicken in F . Then every chicken in F who pecks s others is a king.*

This Corollary answers the question, “Can there be more than one king?” If we have a flock of 3 chickens which peck cyclically, then $s(c)=1$ for each chicken c . Thus all 3 are kings. Perhaps it is not unreasonable for all 3 to be called kings in this case; the flock is small and symmetric. In a large flock, though, we would want it to be unlikely, if not impossible, for all chickens to be kings. Thus we have not really answered the question about the existence of multiple kings until we have comprehensive results about how many kings there can be and how likely it is that there be that many.

Incidentally, for c to be a king it is not necessary that $s(c)$ be maximum. Can you give an example? If there are n chickens, can you determine how small $s(c)$ can be for a king?

The following lemma enables us to prove the next several results more easily. First note that every nonempty subset of a flock is actually a **subflock**; that is, the subset and its pecking pairs form a flock themselves. By picking a certain subflock and applying Theorem 1, we gain useful knowledge about the whole flock.

LEMMA 3. *In a flock, every chicken who is pecked is pecked by a king.*

Proof. Suppose $c \in F$ is pecked, that is, $D_c \neq \emptyset$. Since D_c is a flock, by Theorem 1 it contains a chicken d who is a king of D_c . We will show that d is actually a king of F . It suffices to show that d dominates every chicken in $F - D_c$ in one or two steps. This is true because $F - D_c = S_c \cup \{c\}$, d pecks c , and c pecks all chickens in S_c .

DEFINITION. Chicken c is an **emperor** of flock F if c pecks every other chicken in F .

THEOREM 4. *A flock has exactly one king if and only if that king is an emperor.*

Proof. “If” is clearly true (Why?). As for “Only if,” suppose c were the only king but not an emperor. Then c is pecked. Hence F has another king by Lemma 3.

THEOREM 5. *No flock has exactly two kings.*

Proof. Suppose F has exactly two kings, c and d . Since d is a king, d dominates c in one or two steps. In particular, c is pecked. Thus c is pecked by a king. This must be d . Repeating the proof so far with d and c switched, we conclude that d is pecked by c . But, by definition of a flock, no pair can peck each other.



What Theorem 4 tells us is disappointing. Ideally, we would like a definition of king which implies that the king is always unique. However, we have already seen that Landau’s definition does not assure uniqueness. Thus our next hope is that the definition assures uniqueness often, or at least in many cases where it isn’t intuitively obvious that there’s a best choice. Alas, what Theorem 4 tells us is that the only case in which Landau’s definition gives a unique king is a case where it is already obvious.

Theorem 5, on the other hand, is promising. What is perhaps the least workable form of oligopoly, duopoly, can never occur. Perhaps there are lots of other numbers of kings which

cannot occur. At the least, Theorem 5 prompts us to attack this question more fully. How shall we make this question more precise? Here is one way.

DEFINITION. Let n and k be integers, $n \geq k \geq 1$. We say that F is an (n, k) flock if F has n chickens and exactly k of them are kings. If F has n chickens, it is an n -flock.

The question now becomes, “For which n and k do there exist (n, k) flocks?” We will answer this completely. We begin by answering a special case: “For what n are there (n, n) flocks?” That is, when can every chicken be a king? The answer is surprising—at least it was to me.

THEOREM 6. For every positive integer n except 2 and 4, there exists an n -flock in which every chicken is a king.

Our proof of this follows by induction from the following Lemmas.

LEMMA 7. If there exists an (n, n) flock, then there exists an $(n+2, n+2)$ flock.

LEMMA 8. There is a $(1, 1)$ flock.

LEMMA 9. There does not exist a $(4, 4)$ flock.

LEMMA 10. There is a $(6, 6)$ flock.

The proof of Theorem 6 from the lemmas is by two separate inductions. By Lemmas 7 and 8, one proves the existence of an (n, n) flock for all odd positive integers. By Lemmas 7 and 10 one proves the existence of an (n, n) flock for all even integers ≥ 6 . Finally, $(4, 4)$ flocks are excluded by Lemma 9, hence $(2, 2)$ flocks are excluded by Lemma 7.

Proof of Lemma 7. This beautiful argument is contained in FIGURE 2. Suppose F is an (n, n) flock. Create two new chickens c and d , and let their pecking relations to each other and to the

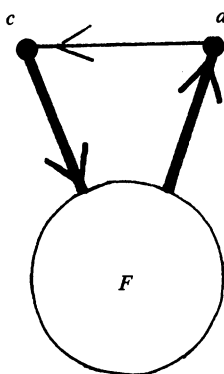


FIGURE 2.

chickens of F be defined as in the figure. Call the augmented flock G . Then c is a king of G (Why?). Also d is a king of G (Why?). Finally, we already know that each $b \in F$ is a king of F . Hence b is a king of G (Why?).

Proof of Lemma 8. This is true vacuously. That is, in any 1-flock the one chicken pecks every other chicken because there are no other chickens.

As for Lemma 9, there are only 64 ways to assign dominance between 4 chickens 2 at a time (Why?). By symmetry the number can be much reduced. In any event, it is clear that the truth value of Lemma 9 can be determined by brute force examination of cases. However, it is esthetic, as well as less tedious, to give a more incisive, conceptual proof. I offer the following sketch. If F is to be $(4, 4)$, no chicken either pecks or is pecked by all the others. Since there are 6

pecking pairs, some two chickens, call them a and b , peck exactly two chickens each, while the remaining two chickens peck exactly one each. One of the first two, say a , uses one of his pecks on the other. These facts uniquely determine F (draw the graph!) and it is not $(4,4)$.

Proof of Lemma 10. To prove this, we need merely display one $(6,6)$ flock. Checking whether a proffered 6-flock works is routine. To devise candidate flocks, it is very helpful to think in terms of graphs. Since all 6 chickens are to be kings, it is reasonable to arrange the arrows as symmetrically as possible. Also, for each c , the chickens that c pecks should be “spread out,” that is, they shouldn’t “waste” their pecks on each other since c must peck all other chickens through them. FIGURE 3 works.

This completes the proof of Theorem 6.

THEOREM 11. *There exist (n,k) flocks for all integers $n \geq k \geq 1$, with the following exceptions: $k=2$ with n arbitrary, and $n=k=4$.*

Proof. We already know that $k=2$ with n arbitrary, and $(4,4)$ are excluded. We must show that nothing else is. The case $k \neq 2,4$ is easy. Start with a (k,k) flock and create $n-k$ new chickens, each of which is pecked by all k old chickens. Arrange the pecking order among the $n-k$ new chickens arbitrarily. Call the augmented flock G . It is easy to see that all the old chickens are still kings in G and that no new chickens are kings in G . Thus G is (n,k) .

For $k=4$, it suffices to show that a $(5,4)$ flock exists, for then an $(n,4)$ flock with $n > 5$ can be created by adding $n-5$ new subservient chickens to the $(5,4)$ flock just as above. A $(5,4)$ flock is shown in FIGURE 4.

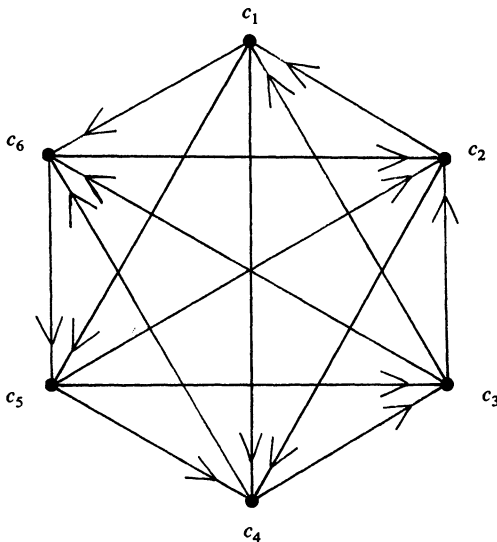


FIGURE 3.

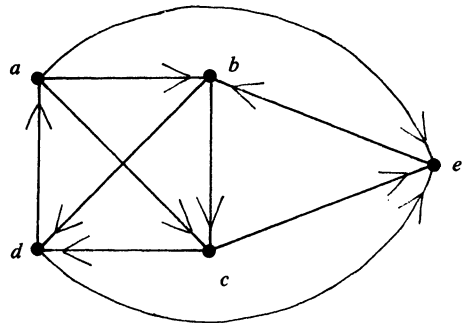


FIGURE 4.

Probabilistic Theorems

The results above about the existence of (n,k) flocks may seem to be evidence enough that our definition of king is not a good one. However, this conclusion is premature. It may be that (n,k) flocks for k large are extremely rare. It may also be that emperors are extremely common. If either of these things is true, the definition would be, in practice, a useful way to determine power in a flock.

How shall we assign probabilities to the various pecking orders a flock can have? With no prior information to go on, it seems reasonable to assign equal probability to each possible pecking pair. That is, for every unordered pair $\{c,d\}$, assign probability $1/2$ to the event that c

pecks d , and $1/2$ to the event that d pecks c . Since an n -flock has $\binom{n}{2}$ unordered pairs, there are $2^{\binom{n}{2}}$ equally likely pecking orders. In what follows we use this probability assumption, and we refer to an n -flock chosen using this assumption as a **random n -flock**. We also use basic rules of probability. In particular, let $\Pr(E)$ be the probability that event E occurs. Let $\{E_i\}$ be a set of events, and let $\cup E_i$ be the event that at least one of the E_i occurs. Then

$$\Pr(\cup E_i) \leq \sum_i \Pr(E_i), \tag{1}$$

with equality iff the E_i are mutually exclusive.

THEOREM 12. *The probability that a random n -flock has an emperor is $n(1/2)^{n-1}$.*

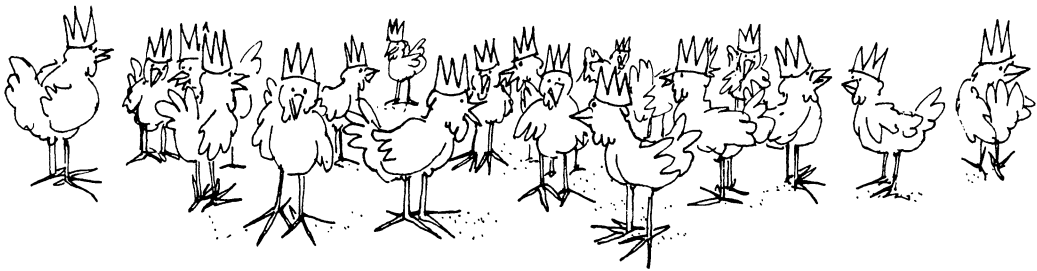
Proof. Let E_i be the event that chicken c_i is emperor. Since no two chickens can be emperors simultaneously, and since by symmetry $\Pr(E_i) = \Pr(E_j)$ for all i and j , the probability we seek is $n\Pr(E_1)$. Moreover, c_1 is emperor iff he pecks all $n-1$ others. Hence $\Pr(E_1) = (1/2)^{n-1}$.

COROLLARY 13. *The probability that a random n -flock has an emperor approaches 0 as $n \rightarrow \infty$.*

Proof. We need one fact from analysis: Exponential functions grow much faster than polynomials. Precisely, if $P(x)$ is any polynomial, and $a > 1$, then

$$\text{as } x \rightarrow \infty, P(x)/a^x \rightarrow 0. \tag{2}$$

One way to prove this is by repeated use of L'Hospital's Rule. Since the probability $p_E(n)$ that a random n -flock has an emperor is $2n/2^n$ (by Theorem 12), it must approach 0 as $n \rightarrow \infty$. (In fact, $p_E(n)$ goes to 0 very fast. For $n \geq 12$, $p_E(n) < .01$.)



That it is very rare for only one chicken to be king did not surprise me, but I was shocked to learn that the opposite extreme—every chicken a king—is very common. I would never have guessed such a thing, let alone proved it, had not a student reported to my class the results of a computer program he ran which counted the number of kings in 100 random flocks of 16 chickens. No flock had fewer than 8 kings, and most had 14, 15, or 16!

THEOREM 14. *The probability that every chicken in a random n -flock is a king approaches 1 as $n \rightarrow \infty$.*

Proof. Let $p(n)$ denote this probability. I do not know an exact expression for $p(n)$, so the proof proceeds by estimates. Let $\bar{p}(n) = 1 - p(n)$ be the probability that *not* every chicken is a king. I will overestimate $\bar{p}(n)$ with a certain computable quantity $q(n)$ and will then show $q(n) \rightarrow 0$ as $n \rightarrow \infty$. Thus $\bar{p}(n) \rightarrow 0$, so $p(n) \rightarrow 1$.

Not every chicken is a king if and only if there exists an ordered pair (c, d) such that c does not peck d and no chicken which c does peck pecks d . Let $p(c, d)$ be the probability that the particular pair (c, d) has this non-dominating property. By symmetry all the $p(c, d)$ are equal. Since there are $n(n-1)$ such ordered pairs, $\bar{p}(n) \leq n(n-1)p(c, d)$ by (1). This last product is the $q(n)$ referred to above. It is easy to compute because $p(c, d)$ is easy to compute. The probability that c does not peck d is $1/2$. The probability that, for a specific third chicken b , c does not

dominate d through b is $3/4$ (Why?). Now, who pecks whom between c and d is independent of their mutual relation to any one b , and their mutual relation to one b is independent of their relation to any other b . Thus $p(c, d) = (1/2)(3/4)^{n-2}$. So

$$q(n) = n(n-1)(1/2)(3/4)^{n-2} = \frac{(8/9)n(n-1)}{(4/3)^n}.$$

By (2), $q(n) \rightarrow 0$. (In fact, $q(n) \leq .01$ when $n \geq 42$. Thus for these same n (and no doubt some lower ones) $p(n) \geq .99$.)

COROLLARY 15. *Let $k(n)$ be the average number of kings in a random n -flock. As $n \rightarrow \infty$, $k(n) - n \rightarrow 0$.*

Proof. Let $p_K(n, i)$ be the probability that a random flock of n chickens has exactly i kings. By definition, $k(n) = \sum_i i p_K(n, i)$. Thus, using the notation $\bar{p}(n)$ as in the proof of Theorem 14,

$$\begin{aligned} n \geq k(n) &\geq n p_K(n, n) = n(1 - \bar{p}(n)) \\ &\geq n \left[1 - \frac{(8/9)n(n-1)}{(4/3)^n} \right] \\ &= n - \frac{(8/9)n^2(n-1)}{(4/3)^n}. \end{aligned}$$

By (2), the latter term goes to 0. Thus $k(n) - n \rightarrow 0$.

The results of this section are much more damaging to our king concept than the previous existence results. Yet, as we point out later, there may still be hope for the definition in terms of its utility for applications to real-world flocks. For now, let us be content that the definition has led to some surprising theorems.

Submissive Chickens and Duality

DEFINITION. Chicken $c \in F$ is a **slave** if c is pecked by all other chickens in F . Chicken c is a **serf** if every other chicken either pecks c or pecks another chicken who pecks c .

Since the definitions of serf and slave are just those of king and emperor “turned around,” it seems clear that all the results about king and emperor go through for serf and slave. The only question is how to justify this without repeating everything.

What we have here is a simple example of **duality**. Duality is a correspondence between concepts so that for every true statement in one’s theory, if one substitutes corresponding concepts throughout, one gets another true statement. For instance, in our theory the correspondence includes: pecks \leftrightarrow is pecked by; king \leftrightarrow serf; emperor \leftrightarrow slave; chicken \leftrightarrow chicken; and flock \leftrightarrow flock. Using this duality (justified in a moment), one gets from Theorem 1, for instance, that every flock of chickens has a serf, and from Lemma 2, that every chicken who pecks pecks a serf.

To justify this duality, note that the fundamental concept in our theory is the pecking order P . Everything else has been defined in terms of this binary relation. For instance a flock is a set C such that for every $c, d \in C$, exactly one of the ordered pairs (c, d) , (d, c) is in P .

Thus we see that every theorem in our theory is really of the form “If P satisfies hypotheses A , then it also satisfies conclusions B .” (Sometimes the only hypothesis is the tacit one that P is the relation of some flock.) Now, the crux is: the name P we have used for the relation is irrelevant. We could substitute any other name P^* in the theorem and the proof would still be valid. Of course any defined terms used in the (usual) statement of the theorem and its proof would have to be replaced by terms defined identically, but using P^* instead of P .

In particular, we could replace $P =$ “pecks” with $P^* =$ “is pecked by.” Making this substitution in the definitions of king and emperor clearly gives serf and slave. Since a chicken is an element in the set on which “pecks” is defined, after the substitution a chicken is still a chicken. Also a

flock is still a flock, for there is no difference in meaning between saying “for every pair, exactly one pecks the other,” and “for every pair exactly one is pecked by the other.” Thus the claimed duality is justified.

In addition to giving a formal boost to one’s theory by providing an analogous theorem for every theorem already proved, duality also gives a psychological boost by suggesting situations in which to look for new types of theorems. Specifically, it gets us to ask about situations in which concepts and their dual concepts interact. For instance, since we know about the probability that every chicken is a king, and dually, about the probability that every chicken is a serf, it occurred to me to ask about the probability that every chicken is both. The answer is another theorem.

THEOREM 16. *Let $p_{KS}(n)$ be the probability that, in a random n -flock, every chicken is both a king and a serf. As $n \rightarrow \infty$, $p_{KS}(n) \rightarrow 1$.*

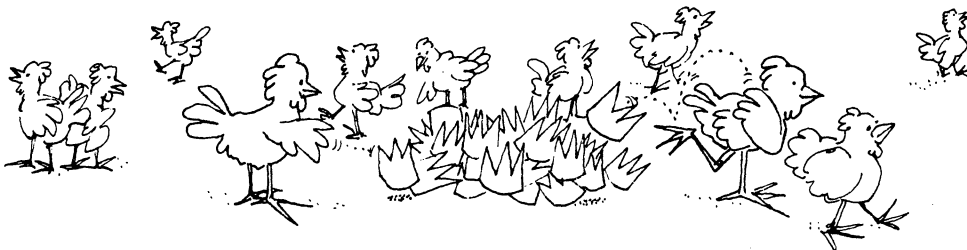
Proof. Let $p_K(n)$ be the probability that all n chickens are kings, and let $p_S(n)$ be the probability that all n are serfs. By Theorem 14, $p_K(n) \rightarrow 1$. By its dual, $p_S(n) \rightarrow 1$. Now, if $E \cap E'$ is the event that events E and E' both occur, then

$$\Pr(E \cap E') \geq \Pr(E) + \Pr(E') - 1. \tag{3}$$

(This is shown by applying (1) to the events not- E and not- E' .) So let E be “all n chickens are kings” and E' be “all n chickens are serfs.” Then (3) becomes $p_{KS}(n) \geq p_K(n) + p_S(n) - 1$. Since $p_{KS}(n) \leq 1$ in any event, the theorem follows.

We ask the reader to prove the next theorem, and to use it to give a much simpler proof of Theorem 16.

THEOREM 17. *If every chicken of F is a king, then every chicken of F is a serf.*



Redefining Kings

Clearly we should consider other definitions of king. I begin with two alternative definitions suggested by students who have heard me lecture on this topic.

According to the current definition, a king dominates every other chicken in one or two steps. Why not allow 3 steps, or 4? To be precise, let us say c dominates d in s steps if there is a sequence of chickens $c = c_0, c_1, \dots, c_{s-1}, c_s = d$ such that for all $i = 0, 1, \dots, s-1$, c_i pecks c_{i+1} . Then c is an **s-king** if c dominates every other chicken in F in at most s steps. In particular, a 1-king is an emperor and a 2-king is a king.

We may now ask: do 3-kings always exist? For which (n, k) do there exist n -flocks with k 3-kings? How often is every chicken a 3-king? Is the definition of 3-king more satisfactory than that of king? What about 4-kings? Etc. Actually, most of these questions are very easy to answer, either by mimicking the proofs of the theorems for kings, or, often enough, by simply using the statements of those theorems. We leave all this to the reader. The upshot is: s -kings are not more satisfactory.

Another approach suggested by students hinges on the idea that, though a flock rarely has just one king, often the set of all kings is at least a proper subset of the flock. Thus, given flock

$F = F_1$, let F_2 be the subflock of all kings of F_1 . In general, for $i \geq 1$, let F_{i+1} be the subflock of all kings of flock F_i . Then F_1, F_2, \dots is a nonincreasing sequence of subsets of F_1 . Since F_1 is finite, there must be a first i for which $F_{i+1} = F_i$; thereafter the sequence is constant. Let us call any c in this F_i a **king-of-kings**.

The idea is that the number of kings-of-kings should be much less than the number of kings. Alas, Theorem 14 tells us that it isn't much less. As soon as n is large enough, most of the sequences F_1, F_2, \dots are constant right from the start. Almost every chicken is a king-of-kings! It also turns out that most of the existential questions we have asked about kings can be answered easily for kings-of-kings too: review the proofs for kings. One intriguing question may appear more difficult: characterize those flocks which have a unique king-of-kings. (A hint on how to answer this question is given later.)

We now turn to definitions of king suggested by studies in the literature of other situations modeled by complete directed graphs. We have already mentioned that such graphs are usually called tournaments. Consider a round-robin tournament among n teams t_1, \dots, t_n . Round-robin means that each pair $\{t_i, t_j\}$ plays exactly once and there are no ties. If we create a vertex i for each t_i , and put an edge from i to j iff t_i beats t_j , then this construction gives us a complete directed graph for each round-robin tournament. Conversely, every complete directed graph represents a possible round-robin tournament.

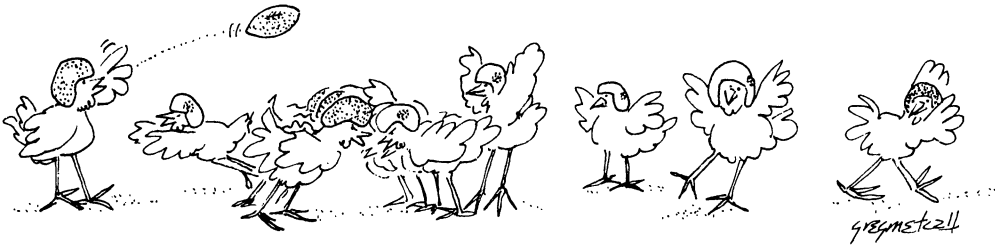
In a real-world tournament, often one wants not only to determine the best team but also to rank them all in descending order. Surely, the basic idea of best team is akin to that of king chicken, so any method for ranking teams ought to have inherent in it a plausible method for choosing a king.

A concept used in many proposed ranking methods is the **score vector**. Let T be a tournament with n teams. Let s_i be the number of victories of team t_i in T . Renumber the teams so that $s_1 \geq s_2 \geq \dots \geq s_n$. Then the score vector is defined to be (s_1, \dots, s_n) . It seems natural to rank the teams in this order, and to declare team t_1 to be the champion. By analogy, we could redefine king to be a chicken who pecks the most others. Indeed, by Corollary 2, this new definition is at least as selective as the original. (What is called s in Corollary 2 is what we now call s_1 .)

However, there may be ties in score vectors. If a tie occurs for top scorer, then this ranking method does not pick a unique king either. In fact, it is possible for *every* team to have the same score. It is not hard to show that this is possible if and only if n is odd.

Another objection is that a team which wins more games may not really be better. What if some t_i wins fewer games than t_1 but wins them all against tougher teams than t_1 wins against? We might be inclined to rate t_i higher than t_1 . Only, how do we judge which teams are tough unless we already have a ranking? Here is one way which has been proposed. We make an initial ranking of the teams based on their score vectors as above. Now, for each team we compute a revised score vector by summing the scores of all the teams it beat. We now have a revised score vector (s'_1, \dots, s'_n) , where it may well be that we have to rerank the teams in order that the revised scores are in descending order. We can repeat the process: Compute a rerevised score for each team by summing the revised scores of all the teams it beat. The rerevised scores may require a further reordering. In theory, we could repeat this process endlessly. If the reader feels intuitively that, after a while, little new information is gained by revising the current score vector again, he is right. It is a theorem that after a finite number of revisions, no further reranking will be caused by further revision. The proof of this uses the theory of eigenvalues of matrices! For precise statements and proofs, see Moon [8] Section 15. Suffice it to say here that this method of choosing a champion (or a king) is not free from objections either.

Another definition of champion is suggested by the following theorem: In every tournament there is a directed path which passes through each vertex once. In terms of chickens in an n -flock, this means it is always possible to number the chickens so that c_1 pecks c_2 , c_2 pecks c_3, \dots , and c_{n-1} pecks c_n . We could redefine a king to be the head of such a chain of command. Unfortunately, this path is rarely unique; in fact, it is unique if and only if the pecking order is



linear. Moreover, as $n \rightarrow \infty$, the probability that in a random n -flock every chicken is a king in this sense goes to 1 too. For more details about this approach, see Roberts [14], Section 3.2. (Roberts does not give a proof of the last, probabilistic assertion. It can be proved using the fact that each so-called strong tournament contains a cycle which passes through all the vertices and the fact that most tournaments are strong. See Moon [8], Theorems 1 and 3.)

Many other methods have been proposed for ranking tournaments, none of them entirely satisfactory. Perhaps this explains why most leagues have special play-offs to determine the champion.

Other Models

Our model produces too many kings, yet changes in the definition of king don't seem to help. So maybe the real problem is with our assumptions about pecking orders rather than with our definitions. To explain this suggestion without undue complications, let us do so within the context of Landau's original definition. We have shown that this definition is unreasonable if we allow a flock to have any pecking order, or even if we assume each pecking order on n chickens is equally likely. But these assumptions were based, to put it politely, on mathematical esthetics and convenience, not on knowledge of chickens. Landau himself was a theoretician of chickens rather than an observer of chickens. But he had read the observational literature. As he notes, although the pecking order in real-world flocks is rarely linear, it is also rarely very far from linear. Thus, for real-world flocks, it may be that his definition works much better than we so far have led ourselves to suspect.

In short, the real problem may be that our assumptions about pecking are too broad. Maybe we should restrict flocks to tournaments which are "almost linear." Maybe we need to use a probability distribution which is very far from random.

Let us pursue the first of these proposed alterations just a bit further. Before we can prove anything about kings in "almost linear" flocks, we need to have a precise mathematical definition of that concept. One way to define it is suggested by the following facts. Proofs may be found in Moon [8], Sections 5-7, but the reader might try to prove them himself. (1) A tournament is linear iff it contains no 3-cycles; (2) the number of 3-cycles in a tournament depends only on the score vector; (3) in a random n -tournament, the expected number of 3-cycles is $(1/4)\binom{n}{3}$. Thus the number of 3-cycles is a measure of how linear a tournament is. According to the observational literature Landau quotes, a typical flock of 10 to 20 chickens contains only 2 or 3 3-cycles. Thus a good topic for further study is this: Are there bounds for the number of kings in terms of the score vector or the number of 3-cycles? If the number of 3-cycles is severely bounded, is the number and nature of the kings severely restricted? If not, what does the number of kings depend on? For more information about the number of 3-cycles as a measure of linearity, see Harary, Norman and Cartwright [2], pp. 300-304. Landau himself proposed a different measure of linearity, which he called the hierarchy index [3, 4].

A word of caution: if the goal is to designate a unique king, then no amount of restriction on pecking orders, either absolute or probabilistic, will cause Landau's definition to be the right one. This is because of Theorem 4. In our present context, what it says is this: Any restriction which leads to Landau kings' being unique is a restriction which makes the concept unnecessary; use the simpler concept of emperor instead.

However, it may well be reasonable to allow a few kings. Then Landau's definition would still be in the running, especially if one can find mathematical restrictions on pecking orders which result in few kings in his sense and which correspond well to the empirical facts about flocks. On the other hand, it may be that the combination of some restrictions on pecking orders and some alternative definition of king will turn out to be best. In any event, this is not something a mathematician can decide by pure cogitation. A thorough knowledge of the facts about chickens is necessary. To pursue this seriously, one might begin by reading the many articles on chickens in the anthology on social hierarchy [15].

Dominance structures occur in many human and animal societies. They were first studied among chickens, where they are especially apparent, quite stable, very hierarchical, and involve every pair; but now many other instances have been documented (see [15] or Chapter 13 of [18]). The point here is that there is considerable variety among species as to the nature of the dominance structure. Mathematical restrictions which are right for chickens may not be right for others. There is room here for a lot of mathematical modeling. Also, a lot of reading of empirical studies and a lot of interaction with social scientists will be necessary if this modeling is to be accurate and useful.

Further Problems

The previous section presents one big open problem. One reason the problem is big is because it is vague; a lot of the work would be in determining appropriate models. Proving theorems would only come later, and might forever be subsidiary, say, to running and evaluating computer simulations.

In contrast, the problems in this section concern our original model and definitions. Thus they are precise and mathematical, but more narrow. Because of the unrealistic implications of this model for kings, these problems are better regarded as pure mathematics than as applied. When I first drafted this article, these problems were all open. Problems 1, 2 and 4 have since been solved, and no doubt more solutions will be found by the time this article appears in print. However, this should not discourage anyone from working on any of them. Different solutions often involve different techniques, are more or less difficult or complete, answer or suggest different generalizations, etc. Problem 4 was solved by an undergraduate. The solution of Problem 1 is announced in [13].

PROBLEM 1. For what 4-tuples (n, k, s, k_s) does there exist an n -flock with exactly k kings and s serfs and such that exactly k_s of the kings are also serfs?

PROBLEM 2. What are the possible structures for subflocks of kings? That is, for which n -flocks F does there exist an m -flock G such that G contains n kings and F is the subflock of kings of G ?

It is not true that every flock F is the king subflock of some G . A necessary condition on F is implicit in Lemma 3. Problem 2 asks for necessary and sufficient conditions. Incidentally, Lemma 3 can be used to answer a question asked in the previous section: When is there a unique king-of-kings?

PROBLEM 3. When do the various definitions of king agree? For instance, when is the set of kings-of-kings the same as the set of kings? When is the set of chickens with the highest score the same as the set of kings? Etc.

PROBLEM 4. Find a proof of Lemma 7 using induction from n to $n+1$ instead of from n to $n+2$. Such an induction can only work starting at $n=5$ (Why?) so it cannot be entirely straightforward.

PROBLEM 5. In a random n -flock, what is the probability that exactly k chickens are kings? That at least k chickens are kings? That at least a particular set S of chickens are kings?

Implicit in our theorems are answers to the “exactly” question for $k=0, 1, 2$ and to the at-least question for $k=0, 1, 2, 3$. The answer to the “set” question is known when S is a singleton. Answers for a few more cases ought to be obtainable by careful brute force, but matters rapidly get complicated: kingships are not independent events. The real question is: are there formulas for general k and S ? Fortunately, if any one of the three questions can be answered completely, the other two follow by the general combinatorial principle of Inclusion-Exclusion. (For discussion of this principle, see any probability or combinatorics book.)

If exact formulas can't be found, what about bounds and asymptotics? We already have an asymptotic result for $k=n$, namely that the probability goes to 1. (Note that the “exactly” and the “at-least” problems are the same for $k=n$.) Thus we automatically have asymptotic results for all $k < n$, namely that the probability goes to 0 for the exactly problem. However, it might be interesting to determine how fast it goes to 0.

PROBLEM 6. In real-world flocks, pecking orders are not permanent. Occasionally the order between two chickens switches. Call one such change a **single switch**. Landau's second paper [4] was devoted to the question of how repeated single switches affect the nature of the pecking order over time, under various assumptions about the nature of those single switches. Landau did not, however, analyze what single switches do to the set of kings. Problem 6 is to analyze this. Specifically, suppose F and G are flocks on the same set of chickens. Let their number of kings be k_F and k_G , where we may assume that $k_F < k_G$. There are many sequences of single switches which turn F into G . Can a sequence be found in which the number of kings always stays the same or increases? If S is the set of chickens who are kings in both F and G , can a sequence be found so that at each state every chicken in S is still a king? These are examples of questions about “minimally disruptive” sequences of single switches from F to G . Still other questions can be made up if one does not specify G . For instance, for every flock F is there a single switch which does not change the set of kings, or at least does not change the number of kings? Does every flock have a single switch which changes the number of kings by at most one? Does every emperorless flock have a single switch which changes the number of kings by exactly one?

Credits

The idea of studying chicken flocks mathematically seems to have originated with Rapoport [10, 11, 12], but the first significant general results are Landau's. Landau succeeded in analyzing the expected value of his “hierarchy index” for arbitrary n -flocks under several different probabilistic assumptions about pecking pairs. As an aside, he proved Theorem 1 and Corollary 2. Landau attributes independent and earlier proofs of Theorem 1 to F. E. Hohn and H. E. Vaughan.

Theorems 4 and 5 appear implicitly in a problem posed in this MAGAZINE by D. L. Silverman [17] and solved there by Moon [9]. The context there is that of debtors in a club instead of henpecked chickens. Lemma 9 appears as half of problem 5 on page 317 of [2]. Theorem 14 is essentially a restatement of part of a result of Moon and Moser, as given by Moon [8], pages 32-34.

It surprises me that more has not been done with Landau's ideas in the 25 years since he wrote [3, 4, 5]. Dominance is an important concept in biology, sociology, and psychology. One wants to know why various groups of living things develop dominance structures, why one particular living thing dominates another, and why dominance structures of whole groups tend to be hierarchical. Landau's hierarchy index is well-known among quantitatively minded social scientists; they compute it as a statistic in analyzing their data. However, little has been done in these 25 years to further develop a mathematical theory of hierarchy formation. Landau himself wrote another paper on this subject in 1965 [6]. He also wrote a survey of all his work on this subject, published posthumously in 1968 [7]. (I recommend this survey as a first paper to read on

this subject.) It seems that the only substantial article to appear since on the mathematics of hierarchy formation is Chase [1].

What have mathematicians done with Landau's ideas? The King Chicken Theorem is only a secondary result in [5]. The main theorem is a characterization of those n -tuples which are the score vectors of tournaments. This is what has interested mathematicians most. This score vector characterization has been reproved several ways and generalized; see Moon [8], Sections 21 and 22.

Landau's king concept seems to have interested mathematicians only to the limited extent already mentioned, and to have interested social scientists not at all. If we limit ourselves to chickens, there are several possible explanations for this lack of interest among social scientists. It has already been noted that the pecking order in chicken flocks is so close to hierarchy that subtle methods of designating top chickens have not seemed necessary. Moreover, the observational literature indicates that even when there is an evident most dominant chicken, this chicken does not *lead* the flock. It merely exercises certain perquisites, e.g., it gets the best food or the best roost. Domestic flocks do not seem to have real leaders.

On the other hand, since the variety of dominance structures in human and animal societies is immense, it is possible that Landau's king concept, or some other mathematical king concept, could still prove useful to social scientists.

In closing, I would like to thank all the mathematicians and students, especially Tim Bock, who have kept me thinking about this subject.

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