ARTICLES

Ramanujan's Notebooks

Working mostly in isolation, Ramanujan noted striking and sometimes still unproved results in series, special functions and number theory.

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We first perceive, then reason later.
Vivekananda

His life spanned but 33 years, and most of these years were lived in virtual obscurity. However, out of such humble beginnings, rose Srinivasa Ramanujan, India's most famous mathematician.

The first purpose of this paper is to give a short account of the life of Ramanujan. While he was in the Indian equivalent of high school, Ramanujan commenced the recording of his mathematical discoveries in notebooks. He continued this practice in the penniless and jobless years which followed. Our second task then is to present a detailed history of Ramanujan's Notebooks from their inception to the time of their publication in 1957. Thirdly, we shall delineate the contents of the Notebooks and describe some of the more fascinating formulas found therein.

Several of the theorems in the Notebooks are rediscoveries by Ramanujan of earlier theorems of others. Some of the results found in the Notebooks were eventually published by Ramanujan, and after his death others, in particular Hardy and Watson, gave proofs of some of the theorems in the Notebooks. In the decade following Ramanujan's death, several papers were published in the Journal of the Indian Mathematical Society that proved results in Ramanujan's Notebooks. However, for the most part, the authors of these papers were unaware that their results were buried in the Notebooks. In particular, the papers of Chowla [13], Malurkar [25], and Rao and Aiyar [43] should be mentioned. It would be an impossible task to reference all material in the Notebooks that is proved elsewhere. However, when known to the author, references to other work will be given. Nonetheless, to this day, many of the discoveries in the Notebooks remain unverified. It is hoped that this paper might stimulate others to investigate this treasure of wondrous formulae and inspirational creativity.

Fuller accounts of the life of Ramanujan may be found in the obituary notices of Aiyar and Rao [1], [44], [40] and Hardy [15], [16], in the lecture of Hardy [20], in the reminiscences of Ranganathan [41], in the Ramanujan Commemoration Volume [9], and in the review of Mordell [26]. For a brief sketch of the early history and another description of the contents of the Notebooks, consult the lecture of Watson [49]. In addition to the Notebooks, Ramanujan left behind some unpublished manuscripts on the partition and Ramanujan tau functions. These manuscripts have recently been examined by Rankin [**42**] and Birch [**10**]. In 1976, George Andrews discovered a "lost notebook" of Ramanujan containing over 100 pages tucked away in the Cambridge University library. This manuscript undoubtedly contains much of the last work of Ramanujan done in the last year of his life.

Life History

Ramanujan was born on December 22, 1887, in Erode, a town in southern India. As was the custom in that time, he was born in the home of his maternal grandparents. Ramanujan's father was a petty accountant to cloth merchants in Kumbakonam about 120 miles east of Erode and about 160 miles south-southwest of Madras. Although he was born into a Brahmin family, his parents were rather poor but not destitute.

Ramanujan's precocity was perhaps first noticed at the age of twelve when he borrowed from an older student a copy of Loney's text on trigonometry. He completely mastered its contents and worked every problem in the book. At the age of sixteen, he borrowed Carr's *Synopsis of Pure Mathematics* from the library of the local Government College. This book was, no doubt, the most significant influence in Ramanujan's mathematical development. The book contains the statements of over 6,000 theorems. There are very few proofs in it, and those proofs that are given are only briefly sketched. The material predates about 1880. It is significant that the book contains no material on elliptic functions, a subject in which Ramanujan became an expert, and nothing on functions of a complex variable, a subject which Ramanujan evidently never learned. In recent years, a later edition of Carr's book [**12**] was reprinted. However, despite being a plentiful source of information, the book is little known and sparingly used today.

In December, 1903, Ramanujan took the matriculation exam of the University of Madras and obtained a "first class" place. However, due to his complete absorption into mathematics and his failure to study English and physiology, a subject which he disliked intensely, Ramanujan failed the exam at the end of his first year at the Government College in Kumbakonam. His parents did not wish their son to devote so much time to mathematics, and it is reported that he did much of his mathematics under a cot in an attempt to escape the eyes of his parents [**41**, p. 86]. He later entered Pachaiyappa's College in Madras but, in 1907, again failed his examination.

From 1907 to 1909 Ramanujan wandered about the countryside impoverished and unemployed, but he continued to do mathematics and record his results in his Notebooks. In 1909, Ramanujan married; his wife Srimathi Janaki recently received the first copy of a commemorative volume [**9**] dedicated to her late husband. In 1910, Ramanujan met V. R. Aiyar, the founder of the Indian Mathematical Society. At that time, Aiyar was a deputy collector in the Madras civil service, and Ramanujan asked him for a job in his office. After perusing the theorems in Ramanujan's Notebooks, Aiyar wrote P. V. Seshu Aiyar, Ramanujan's mathematics instructor while at the Government College in Kumbakonam. Seshu Aiyar, in turn, wrote to another mathematician R. Ramachandra Rao who arranged a meeting with Ramanujan. The contents of Ramanujan's Notebooks exceedingly excited Rao, and he gave Ramanujan a monthly allowance so that Ramanujan could continue his work on mathematics. Thus, for the first time, Ramanujan could devote his full energies to mathematics without the burden of worrying about money for daily subsistence. However, Ramanujan was bothered by the fact that he was receiving money without really formally occupying a job. Thus, on March 1, 1912, Ramanujan became a clerk in the Madras Port Trust Office. This turned out to be most fortuitous, for the chairman of the Madras Port Office was Sir Francis Spring who took a great interest in Ramanujan and his mathematical work.

Spring was a prominent engineer who knew a fair amount of mathematics, and the manager S. N. Aiyar of the Madras Port Office was a well-known mathematician. They, along with P. V. Seshu Aiyar, encouraged Ramanujan to write to the famed English mathematican G. H. Hardy, and on January 16, 1913, Ramanujan sent the first of his now famous letters to Hardy. Hardy [**40**, p. xxii] later indicated that he felt that Ramanujan had considerable help in drafting the letter because

Ramanujan's facility with English was lacking. Indeed, apparently the letter was composed, for the most part, by S. N. Aiyar and Sir Francis Spring [**9**, p. 47]. The letter and two subsequent letters to Hardy that followed shortly thereafter contained the statements of approximately 120 theorems. On receiving the first letter, Hardy dismissed the results contained therein as the work of a crank. However, on that evening, he and Littlewood examined the letter with considerably more care, and, despite the falsity of some of the results, they were astonished with many of the findings. Hardy wrote to Ramanujan with the strong suggestion that he come to Cambridge so that his already great talents could be further developed. Because of strong caste convictions and the refusal of his mother to give permission, Ramanujan declined Hardy's invitation. Further, partly through the persuasion of Sir Gilbert Walker, on March 19, 1913, Ramanujan was elected to a scholarship at Madras. Thus, on May 1, Ramanujan began an academic career that would allow him to devote his full resources to mathematics without going to England.

At the beginning of 1914, the Cambridge mathematician E. H. Neville sailed to India to lecture in the winter term at the University of Madras. One of Neville's principal tasks was to convince Ramanujan to come to England. Probably more important than the persuasions of Neville were a pilgrimage of Ramanujan and a dream of his mother. According to N. Subbanarayanan [**9**, p. 48], his father, S. N. Aiyar, and Ramanujan made a trip to Namakkal where they stayed in the temple of Goddess Namagiri three days and three nights. On the last night, Ramanujan received the command from Goddess Namagiri to go to England. His mother had a dream in which she saw her son surrounded by Europeans. Goddess Namagiri then told her to allow her son to depart for England. Thus, on March 17, 1914, Ramanujan left Madras for England. The departure was very difficult for Ramanujan. He was persuaded to cut off his tuft of hair which had religious significance to him. The unaccustomed wearing of European clothes and shoes made him quite uncomfortable and did not mitigate his anxiety.

On April 14, Ramanujan reached London. (Historical accounts [**9**, pp. 43, 72, 109], [**41**, pp. 70–71] differ as to whether Ramanujan was accompanied on the boat by Neville or whether Neville only met Ramanujan at the dock.) In the three years that followed, Ramanujan was very active mathematically. He talked with Hardy almost daily, and each profited immensely from the stimulation of the other. Much of Ramanujan's best work was done in collaboration with Hardy. However, Ramanujan found it difficult to adjust to English weather. Being a strict vegetarian, he cooked for himself. Due partly to the war, he could not always obtain suitable food sent to him from India. In May 1917, Hardy wrote that Ramanujan had apparently contracted an incurable disease which in the following year was diagnosed as tuberculosis. The war prevented Ramanujan from immediately returning to India, but on February 27, 1919, Ramanujan departed from England. The more favorable weather and diet of India, however, did not abate Ramanujan's illness, and on April 26, 1920, he died.

History and general content of Ramanujan's Notebooks

In 1903, or perhaps earlier, while in school, Ramanujan began to collect his theorems in notebooks. He added to his notebooks until about 1913. There are two main Notebooks. The second is a revision of the first and contains additional material as well. The second edition might have been started while Ramanujan held a scholarship at Madras. There is also a very short third Notebook which is of a fragmentary nature. Together, the three Notebooks contain the statements of approximately 3000–4000 theorems. Emil Grosswald has suggested the possibility that Ramanujan left us four notebooks in that the second Notebook, in reality, contains two separate notebooks. In the last 53 pages of what is commonly designated as Notebook 2, Ramanujan commenced a new page numbering system. The Notebooks which survive are probably not the earliest notebooks in which Ramanujan recorded his results. In 1912, Sir Gilbert Walker saw 4 or 5 notebooks of Ramanujan, each about an inch thick and each in a black cover.

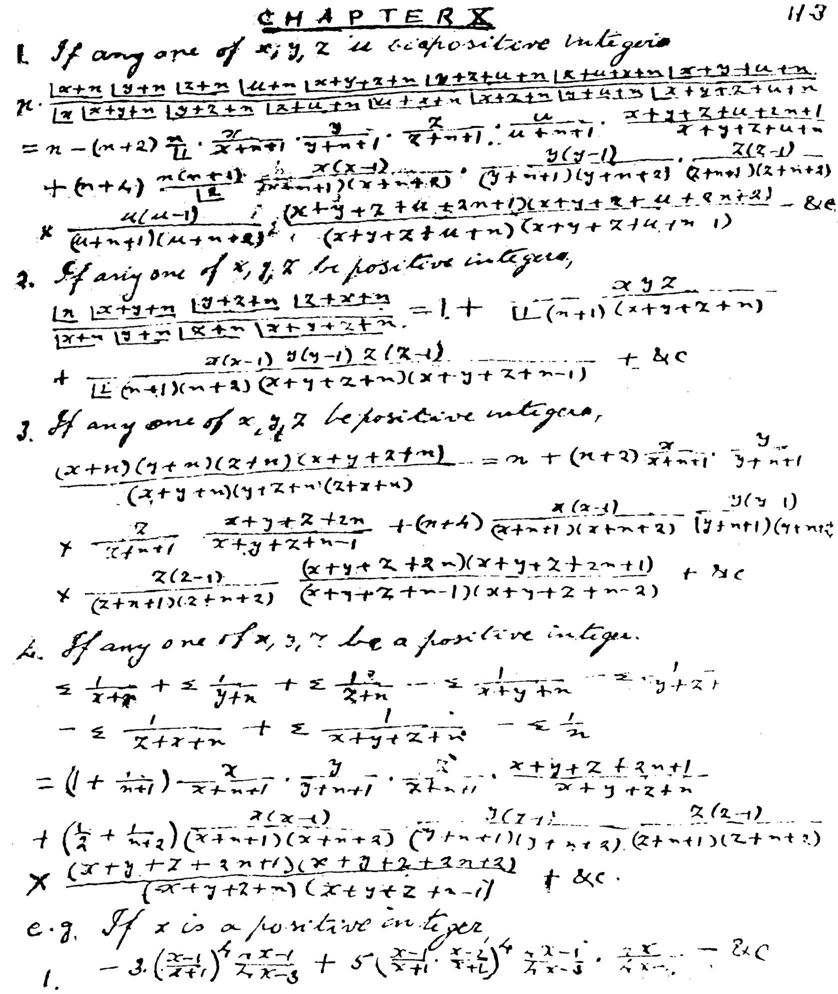
The first of Ramanujan's Notebooks was left with Hardy when Ramanujan left England in 1919. The second and third Notebooks were taken back to India and were acquired by the University of Madras on Ramanujan's death. Apparently, in India, the location of the first Notebook was not

Srinivasa Ramanujan

known at that time. In 1924, S. R. Ranganathan, a mathematician and the University librarian at Madras, journeyed to Cambridge to study library science for a year. While in England, he visited Hardy who was now at Oxford. Hardy gave Ranganathan the first Notebook and strongly urged the publication and editing of the Notebooks. Ranganathan took the Notebook back to Madras where three handwritten copies of all three Notebooks were made. One of the handwritten copies was sent to Hardy.

With the assistance of B. M. Wilson, G. N. Watson undertook the task of editing the Notebooks in 1929. At that time, he estimated that it would take them at least five years to accomplish the task. Partly due to the premature death of Wilson, the task of editing the Notebooks was never completed. However, Watson wrote approximately 25 papers inspired by theorems in the Notebooks, especially the latter parts of the second Notebook. Thus, more than any other mathematician, Watson brought some of the contents before the mathematical public even though he never realized his initial aim of editing the entire Notebooks.

Late in the first half of this century, interest in publishing and editing the Notebooks was revived. In 1949, three photostat copies of the Notebooks were made at the University of Madras, and two of


CHAPTER X 113

1. If any one of x, y, z is a positive integer

2. If any one of x, y, z be positive integers,

3. If any one of x, y, z be positive integers,

4. If any one of x, y, z be a positive integer.

e.g. If x is a positive integer,

A typical page from the second volume of Ramanujan's notebooks.

them still remain in the library there. The third copy was given to the noted Indian mathematician S. S. Pillai. In 1950, Pillai lost his life in a plane crash in Egypt while enroute to the International Congress of Mathematicians in Cambridge, Massachusetts. His copy of the Notebooks was with him.

In 1954, the publishing of the Notebooks was suggested at the meeting of the Indian Mathematical Society in Delhi. Finally, in 1957, the Tata Institute of Fundamental Research in Bombay published in two volumes a photostat copy of the Notebooks edited by K. Chandrasekharan. The first volume reproduces Ramanujan's first Notebook, while the second reproduces his second and third Notebooks. Only 1000 copies of the Notebooks were published. The facsimile edition contains no commentary whatsoever and is undoubtedly far less than the project originally envisioned by Hardy and Watson. The reproduction, however, is very faithful and clear. If one side of a page is left blank by Ramanujan in the Notebooks, it is left blank in the facsimile edition. Ramanujan's "scratch" work is also faithfully reproduced. Thus, on one page, we find only the fragment, "If r is positive." The reproduction was done on heavy, large pages with very generous margins. The author weighed the two volumes on his bathroom scales, and they weighed together over ten pounds. On the other hand, the excellent reproduction on such sturdy paper will insure their preservation for the years to come, and the mathematical community should be very grateful to the Tata Institute for performing such a valuable service.

The first of Ramanujan's Notebooks was written in a peculiar green ink. The book has 16 chapters containing 134 pages. Following these 16 chapters are approximately 80 pages of heterogeneous material. At first, Ramanujan wrote on only one side of the page. He later began using the back side of the pages for miscellaneous "scratch" work, which, for the most part, had no connection with the remainder of the material in the chapter. The chapters are somewhat organized into topics, but often there is no apparent connection between adjacent sections of material in the same chapter. Very seldom is there any indication of a proof or verification of a theorem.

The second Notebook contains 21 chapters with a total of 252 pages. These chapters are followed by 100 pages of heterogeneous formulas. In contrast to the first Notebook, Ramanujan used both sides of the paper in the second Notebook. Brief proofs are somewhat more frequent in the second Notebook than in the first, but by no means are they very numerous. Many people have conjectured about the nature of Ramanujan's proofs. His proofs were undoubtedly a mixture of intuition, examples, and induction, but evidently little more can be said. The successive listing of theorems with few verifications was clearly influenced by the style of Carr's *Synopsis*.

Chapters 1 and 2 of each volume cover, respectively, the same topics. Chapter 3 of the second Notebook contains material from Chapters 3 and 4 of the first. Chapter 4 of the second Notebook has material from Chapter 5 of the first. The material from Chapters 6 and 7 of the first is found in Chapter 5 of the second. Chapter n, $6 \leqslant n \leqslant 12$, of the second Notebook roughly corresponds to Chapter $n+2$ of the first, although material from Chapter 12 of the first Notebook can be found in both Chapters 10 and 11 of the second. Contents of Chapter 15 of the first are found in Chapters 16 and 17 of the second, and material from Chapter 16 of the first is in Chapter 14 of the second. Chapters 13, 15 and 18–21 of the second Notebook contain largely material not found in the first Notebook.

The third Notebook contains but 33 pages of miscellaneous work and is not divided into chapters. In this Notebook are found some of Ramanujan's results in number theory. There is an interesting story [**41**, p. 16] in connection with a table of natural numbers found on pages 8 and 9 of the third Notebook (pp. 368–369 of [**39**, vol. II]). In 1934, over a ouija board, Ranganathan and a friend of Ramanujan in his youth, K. S. Krishnaswami Ayyangar, invoked the spirit of Ramanujan. In replying to a question concerning the continuation of his mathematics, Ramanujan remarked that "All interest in mathematics dropped out after crossing over." Ramanujan was also queried about the meaning of the aforementioned table of numbers. He could not remember anything about the table. However, having been invoked in another seance one week later, Ramanujan informed that the table was related to his work on mock theta functions. This seems very unlikely. First, the work in the Notebooks was done before Ramanujan left for England, while the theory of mock theta functions was Ramanujan's last contribution to mathematics before he died. Secondly, the table consists of all natural numbers through 12,005 whose factorizations contain only the primes 2, 3, 5, and 7, and so has no connection with mock theta functions, but instead with Ramanujan's work on highly composite numbers [**37**].

Before giving some specific examples from the Notebooks, we shall make a few general remarks on their contents. Ramanujan clearly loved infinite series. In the formal manipulation of series, possibly only Euler and Jacobi were his equals. Many identities involving infinite series found in his Notebooks remain unverified to this day. Often the series do not converge, and seldom was there any mention of this. He discovered many methods of summability, including those of Abel and Borel, but Ramanujan made no distinction between ordinary convergence and summability. However, he did have a rather peculiar theory of divergent series. Ramanujan seldom used the summation sign Σ and instead preferred to write out the first few terms of the series followed by the symbol &c. Occasionally, he would write, for example, $\Sigma\ 1/n$ with no indices of summation. The only other instances of the employment of the summation sign are in double summations where the summation sign is used to denote the outer summation. At times, Ramanujan's infrequent discourse on series is obscure. For example, on page 181 of volume 2, Ramanujan states that "... $F(h)$ also will be an infinite series; but if most of the numbers p,q,r,s,t &c be odd integers $F(h)$ appears to terminate. In this case the hidden part of $F(h)$ can't be expanded in ascending powers of h..." It is not always clear why Ramanujan considered various functions defined by infinite series. Intriguing series identities are produced throughout the Notebooks. Often, changes of variables and the specialization of parameters yield beautiful results. There are many evaluations of series which are not found in any of the standard tables. Many of these identities and evaluations, as well as other theorems of Ramanujan would serve as interesting problems for the MONTHLY.

Several of Ramanujan's intriguing series identities resulted from adroit applications of variants of the Poisson summation formula and various other types of summation formulas. With suitable restrictions on f, the Poisson summation formula may be written as

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\cos(2\pi nx)\,dx.$$

This, the Abel-Plana summation formula, and several new summation formulas were discovered by Ramanujan. He never stated conditions under which the formulas were valid. Various versions of the Euler-Maclaurin summation formula were skillfully used by Ramanujan in evaluating series, in obtaining series relations, and in approximating sums. For recent accounts of the Euler-Maclaurin formula and its applications, see the papers of Boas **[11]** and Berndt and Schoenfeld **[8]**.

Ramanujan was also quite fond of the Bernoulli numbers B_n which are customarily defined today by

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \qquad (|x|<2\pi).$$

At least four different conventions for the Bernoulli numbers are found in the literature, and Ramanujan used at least two of them in the Notebooks; the reader must be careful about this. Many of the formulae that Ramanujan derived involve Bernoulli numbers. Ramanujan's first published paper **[30]**, **[40**, pp. 1–14] is on Bernoulli numbers. Ramanujan calculated several of the Bernoulli numbers and had an excellent facility for calculation, although he was not as exceptional as many others.

Although many unusual integrals, several involving infinite products, are evaluated in the Notebooks, Ramanujan's contributions in this area are not nearly as numerous or as profound as his contributions to infinite series. Several beautiful identities between integrals arose from a skillful use of Parseval's identity for integrals or from similar theorems, all of which Ramanujan discovered independently.

A major portion of the latter half of Ramanujan's second Notebook is devoted to formulae from the theory of elliptic modular functions. In this area, Ramanujan is surpassed by no other. Watson **[49]** remarked that "A prolonged study of his modular equations has convinced me that he was in possession of a general formula by means of which modular equations can be constructed in almost terrifying numbers."

Ramanujan's continued fraction expansions of functions are striking. Among the theorems enounced in Ramanujan's first letter to Hardy, Hardy found those on continued fractions perhaps the most interesting.

Ramanujan used the old-fashioned notation $\underline{n|}$ for $n!$. He also used the notation $\underline{n|}$ when n is not a positive integer, where we, of course, would denote $\underline{n|}$ by the gamma function $\Gamma(n+1)$. As customary in his day, Ramanujan denoted the Riemann zeta-function $\zeta(n)=\sum_{k=1}^{\infty}k^{-n}$, $n>1$, by S_n. The reader should be warned, however, that the notation S_n was used by Ramanujan in other contexts as well.

In a certain sense, mathematics has been advanced most
by those who are distinguished more for intuition
than for rigorous methods of proof.
—Felix Klein

Some fascinating entries from the Notebooks

Since the second Notebook is an expanded version of the first, we shall make most of our references to the second Notebook. The indicated pagination will always be that of the edition published by the Tata Institute. Formulas are quoted as Ramanujan wrote them, and no attempt has been made to verify them. For economy of space, we shall frequently use the summation sign Σ instead of Ramanujan's notation.

A magic square is a square array of natural numbers such that the sum of the elements in any column, row, or diagonal is the same. For several centuries, magic squares have been popular in elementary Indian mathematics. Ramanujan's very early preoccupation with mathematics focused on magic squares. In the opening chapter of each Notebook, Ramanujan gives details on how to construct magic squares and other square arrays satisfying certain properties. Details are much fuller in the second volume than in the first. In [**9**, p. 100], M. Venkataraman gives the following magic square:

22	12	18	87
21	84	32	2
92	16	7	24
4	27	82	26

constructed according to directions found in the Notebooks. Note that the first row gives the date of Ramanujan's birth. For excellent accounts of magic squares, consult the books of Ball and Coxeter [**5**] and W. S. Andrews [**3**].

In the second Notebook, Chapter 1 is prefaced by two pages of columns of natural numbers. The numbers in the columns at the right are highly composite numbers, i.e., numbers with many small prime factors. The columns at the left give the number of divisors of each of these highly composite numbers. One of Ramanujan's most significant papers [**37**], [**40**, pp. 78–128] is concerned with this topic.

Chapter 2 is largely devoted to sums involving the reciprocals of integers. On page 7 of the first Notebook we find the following interesting derivation. Ramanujan first proves that

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=\frac{n}{2n+1}+\frac{1}{2^3-2}+\frac{1}{4^3-4}+\cdots+\frac{1}{(2n)^3-2n}. \tag{1}$$

He then lets $n=1/dx$ in the left side of (1). From

$$\frac{2dx}{1+dx}+\frac{2dx}{1+2dx}+\cdots+\frac{2dx}{1+1}=2\int_1^2\frac{dx}{x}=2\log 2 \tag{2}$$

and (1), upon letting n tend to ∞, Ramanujan deduces that

$$1+\sum_{n=1}^{\infty}\frac{2}{(2n)^3-2n}=2\log 2.$$

Of course, the first equality in (2) is not quite correct. Ramanujan is really approximating the integral on the right by a Riemann sum. The first problem that Ramanujan published in the Journal of the Indian Mathematical Society is a very similar result [**28**], [**40**, p. 322].

Chapter 2 also contains some identities that are useful in obtaining approximations of partial sums of the harmonic series. Thus, on page 17 of the second Notebook, Ramanujan says that

$$\sum_{n=1}^{1000} \frac{1}{n} = 7\tfrac{1}{2} \text{ very nearly.}$$

Ramanujan was quite fond of using the expressions "nearly" and "very nearly".

Ramanujan had a proclivity for sums involving $\tan^{-1}x$, and in Chapter 2, we find several such finite sums. As an example, we quote (vol. II, p. 16)

$$\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\cdots+\tan^{-1}\frac{1}{2n}+\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}+\cdots+\tan^{-1}\frac{1}{4n+1}$$
$$=\frac{\pi}{4}+\tan^{-1}\frac{9}{53}+\tan^{-1}\frac{18}{599}+\cdots+\tan^{-1}\frac{9n}{32n^4+22n^2-1}$$
$$+\tan^{-1}\frac{4}{137}+\tan^{-1}\frac{8}{2081}+\cdots+\tan^{-1}\frac{4n}{128n^4+8n^2+1}.$$

Other topics in Chapter 2 include the infinite product representations for sin x and cos x and related infinite products, as well as the approximation of roots of polynomials by convergents of continued fractions.

At the beginning of Chapter 3 of the first Notebook, we find the evaluation

$$\int x^{n-1}e^x\,dx = e^x\left\{\int x^{n-1}dx-\int\int x^{n-1}(dx)^2+\int\int\int x^{n-1}(dx)^3-\&\text{c}\right\}$$
$$=e^x\left\{\frac{x^n}{n}-\frac{x^{n+1}}{n(n+1)}+\frac{x^{n+2}}{n(n+1)(n+2)}-\&\text{c}\right\}.$$

The above is merely a rather unorthodox way of writing the successive integrations by parts for $x^{n-1}e^x$. The main content of Chapter 3, however, is concerned with Taylor series and various other types of expansions of functions, many related to the exponential function.

Ramanujan begins Chapter 4 of the second Notebook (Chapter 5 of the first) with a study of the functions $F_n(x)$ defined recursively by

$$F_1(x)=e^x-1, \quad F_{n+1}(x)=e^{F_n(x)}-1, \qquad n\geqslant 1.$$

There follow several beautiful formulas on integral transforms of which

$$\int_0^\infty x^{n-1}\{\varphi(0)-x\varphi(1)+x^2\varphi(2)-\&\text{c}\}\,dx=\frac{\pi\varphi(-n)}{\sin(\pi n)}$$

is a striking example (vol. II, p. 45). This formula as well as other work of Ramanujan on Fourier transforms is discussed by Hardy in [19]. Parseval's theorem for integrals and the inversion formulae for Fourier cosine transforms are given (vol. II, p. 46). Several Maclaurin series in $1/x$ are "evaluated" at $x=0$. Thus, for example, Ramanujan states that (vol. II, p. 44) "When $x=0$

$$\frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}-\&\text{c}=\frac{\pi}{2} \tag{3}$$

which is same as saying $\tan^{-1}\infty=\pi/2$." Of course, the series on the left side of (3) is simply the Maclaurin series of $\tan^{-1}(1/x)$.

Chapter 5 of the second Notebook opens with some formulae that resemble the Euler-Maclaurin formula. The chapter is primarily devoted to the Bernoulli numbers B_n and various series expansions involving the Bernoulli numbers. Several properties of Bernoulli numbers are listed. On page 57, we find Euler's formula for $S_{2n}=\zeta(2n)$ and similar formulae. The values $S_2,\ldots,S_{10}$ are calculated to 10 decimal places. However, Legendre had long ago calculated $S_2,\ldots,S_{35}$ to 16 decimal places, and Stieltjes had calculated $S_2,\ldots,S_{70}$ to 32 decimal places [47]. Several striking series are evaluated in terms of S_n (pp. 59–61). For example, for $n\geqslant 2$,

$$\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{8^n}+\frac{1}{11^n}+\frac{1}{12^n}+\&\text{c}=\frac{S_n^2-S_{2n}}{2S_n},$$

where the sum is over those natural numbers containing an odd number of prime factors in their factorizations. Most of these sums are contained in [31], [40, pp. 20–21]. Bernoulli numbers of fractional index are defined by generalizing Euler's formula for S_{2n}. In other words, Ramanujan defines B_n, where $n>1$ is real, by $B_n=2\lfloor n\, S_n/(2\pi)^n$. The Euler numbers E_{2n} may be defined by

$$\sec x=\sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}}{(2n)!}x^{2n}\qquad (|x|<\pi/2),$$

and Ramanujan develops a theory of Euler numbers analogous to his theory of Bernoulli numbers.

Chapter 6 of the second Notebook (Chapter 8 of the first) is, on a first examination, perhaps the most mysterious chapter in the Notebooks. Here, Ramanujan expounds his strange theory of divergent series. This theory depends upon a version of the Euler-Maclaurin summation formula wherein the number of approximating terms involving Bernoulli numbers is taken to be infinite. In Ramanujan's theory, a "constant" is associated to certain divergent series. He concludes his discussion of the "constant" with the observation (vol. I, p. 79), "The constant of a series has some mysterious connection with the given infinite series and it is like the centre of gravity of a body. Mysterious because we may substitute it for the divergent infinite series." He then gives some examples. The "constant" for $1+1+1+1$ &c is said to be $-1/2$. He later defines a series to be "corrected" when its "constant" is subtracted from it. Ramanujan's theory of divergent series has been placed on a firm foundation by Hardy [22, Section 13.15]. However, he does not mention the connection between his theory and Ramanujan's discourse in the Notebooks.

Chapter 7 in the second Notebook is a continuation of Chapter 5 of the same Notebook. There are several theorems concerning $\varphi_r(x)=1^r+2^r+\cdots+x^r$. Bernoulli numbers of negative index are defined by means of the functional equation of the Riemann zeta-function $\zeta(s)$ for real values of s, given on page 78 of the second volume. Similarly, the functional equation of $\sum_{k=1}^{\infty}(-1)^k(2k+1)^{-s}$, $s>0$, is given on page 86 in the notation of Euler numbers, and hence this equation is used to define Euler numbers of negative index. The chapter concludes with several standard properties of the gamma function, including Stirling's formula.

Chapter 10 of the first Notebook and Chapter 8 of the second do not correspond very well. In particular, Chapter 10 contains material not found in the revised edition. Euler's constant, usually written in decimal notation, is found in several places throughout the Notebooks, although it is never given any name. In Chapter 10 of the first Notebook, we find its decimal expansion .57721566490153286060..., which is correct to the number of decimal places given. Several series are evaluated in terms of logarithms. One of the simpler examples is (vol. II, p. 94)

$$\frac{\sqrt{3}-1}{1}-\frac{(\sqrt{3}-1)^4}{4}+\frac{(\sqrt{3}-1)^7}{7}-\&\text{c}=\frac{\pi}{4\sqrt{3}}+\frac{1}{3}\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right).$$

Ramanujan also evaluates some integrals including

$$A_n=\int_0^x\frac{dx}{1+x^n}$$

for $n=1-6$, 8, and 10. The formula for A_3 (vol. II, p. 94) can be used to derive the series evaluation given immediately above. The formula

$$A_5=\frac{1}{20}\log\frac{(1+x)^5}{1+x^5}+\frac{1}{4\sqrt{5}}\log\frac{1+x\dfrac{\sqrt{5}-1}{2}+x^2}{1-x\dfrac{\sqrt{5}-1}{2}+x^2}$$

$$+\frac{1}{10}\sqrt{10-2\sqrt{5}}\,\tan^{-1}\frac{x\sqrt{10-2\sqrt{5}}}{4-x(\sqrt{5}+1)}+\frac{\sqrt{10+2\sqrt{5}}}{10}\tan^{-1}\frac{x\sqrt{10+2\sqrt{5}}}{4+x(\sqrt{5}-1)}$$

should convey to the reader an appreciation of Ramanujan's stamina for computation. On page 97 of the second volume, we find the highly unusual product evaluation

$$\frac{\left(\frac{\sqrt[1]{1}}{\sqrt{2}}\cdot\frac{\sqrt[3]{3}}{\sqrt[4]{4}}\cdot\frac{\sqrt[5]{5}}{\sqrt[6]{6}}\cdot\frac{\sqrt[7]{7}}{\sqrt[8]{8}}\&\text{c}\right)^{1/\log 2}}{\left(\frac{\sqrt[1]{1}}{\sqrt[3]{3}}\cdot\frac{\sqrt[5]{5}}{\sqrt[7]{7}}\cdot\frac{\sqrt[9]{9}}{\sqrt[11]{11}}\cdot\frac{\sqrt[13]{13}}{\sqrt[15]{15}}\&\text{c}\right)^{4/\pi}}=\frac{\sqrt{2}}{\pi}\left(\underline{|-1/4}\right)^4.$$

Ramanujan also remarks (vol. II, p. 95) that "$\underline{|x}$ is minimum when $x=6/13$ very nearly." Since $6/13=.46153\ldots$, this is very close to the actual minimum $.46163\ldots$.

The next chapter is largely devoted to trigonometric series and contains some of the most interesting formulas found in the Notebooks. On page 145 of volume 1, we find that

$$\cos 2x-\left(1+\frac{1}{3}\right)\cos 4x+\left(1+\frac{1}{3}+\frac{1}{5}\right)\cos 6x-\&\text{c}$$
$$=\frac{\pi}{4}(\cos x-\cos 3x+\cos 5x-\&\text{c}),$$

but neither side converges. Some interesting expansions in powers of trigonometric functions are found. This next intriguing relation (vol. II, p. 112)

$$\sum_{n=1}^{\infty}\frac{1}{n^2}(x^n\cos n\theta+y^n\cos n\varphi)$$
$$=\frac{1}{8}\log(1-2x\cos\theta+x^2)\log(1-2y\cos\varphi+y^2)-\frac{1}{2}\tan^{-1}\frac{x\sin\theta}{1-x\cos\theta}\tan^{-1}\frac{y\sin\varphi}{1-y\cos\varphi}$$

is supposedly valid under the assumptions $x\cos\theta+y\cos\varphi=xy\cos(\theta+\varphi)$ and $x\sin\theta+y\sin\varphi=xy\sin(\theta+\varphi)$. Many relations involving the dilogarithm $\sum_{n=1}^{\infty}x^n/n^2$, $|x|\leqslant 1$, are listed with no hints of verification. Proofs of most of these relations and many other properties of the dilogarithm may be found in Lewin's treatise [**24**].

Chapters 10 and 11 of the second Notebook, or Chapters 12 and 13 of the first, develop the theory of hypergeometric series. These chapters have been thoroughly examined by Hardy [**17**]. Ramanujan independently derived practically all of the major classical theorems in the subject, including theorems of Gauss, Kummer, Dougall, Whipple, Dixon, Saalschütz, Schäfli, Clausen, and Thomae. In most instances, Ramanujan discovered the theorems later than those whose theorems bear their names. For proofs of most of the theorems in these two chapters, see Hardy's paper [**17**] or Bailey's tract [**4**]. Hardy [**18**] has also pointed out that many series summed by Ramanujan can be evaluated by using theorems about hypergeometric series.

Continued fraction expansions are discussed in the next chapter. We cite just two formulae. The first (vol. II, p. 147)

$$\sqrt{\frac{2x}{\pi}}-\frac{x}{1+}\,\frac{2x}{2+}\,\frac{3x}{3+}\,\frac{4x}{4+}\&\text{c}=\frac{2}{3\pi}\quad\text{when}\quad x=\infty$$

is an example to illustrate a theorem which "is only approximately true." The second (vol. II, p. 149)

$$\frac{\underline{\left|\frac{x+m+n-1}{2}\right.}\;\underline{\left|\frac{x-m-n-1}{2}\right.}-\underline{\left|\frac{x+m-n-1}{2}\right.}\;\underline{\left|\frac{x-m+n-1}{2}\right.}}{\underline{\left|\frac{x+m+n-1}{2}\right.}\;\underline{\left|\frac{x-m-n-1}{2}\right.}+\underline{\left|\frac{x+m-n-1}{2}\right.}\;\underline{\left|\frac{x-m+n-1}{2}\right.}}$$

$$=\frac{mn}{x+}\,\frac{(m^2-1^2)(n^2-1)}{3x+}\,\frac{(m^2-2^2)(n^2-2^2)}{5x+}\,\frac{(m^2-3^2)(n^2-3^2)}{7x+\&\text{c}}$$

is one of many similar continued fraction expansions given by Ramanujan. Several of the continued fraction expansions appear to be related to hypergeometric functions. Some of Ramanujan's theorems bear a resemblance to continued fraction expansions found in the book of Khovanskii [23]. However, most of the results in this chapter apparently have not been proved in print.

Chapter 13 of the second Notebook concentrates on theorems akin to the Fourier integral theorems, on theorems similar to Parseval's theorem, and on the evaluation of definite integrals. Most of the results are not difficult to prove. As an example, if $\alpha\beta=\pi$, then (p. 160)

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}}{e^{\alpha x}+e^{-\alpha x}}\,dx=\sqrt{\beta}\int_0^\infty \frac{e^{-x^2}}{e^{\beta x}+e^{-\beta x}}\,dx.$$

This identity was one of the first of several questions that Ramanujan published in the Journal of the Indian Mathematical Society [29], [40, p. 324]. See also [35], [40, pp. 53–58]. The Poisson summation formula (p. 165) and other summation formulas are found in this chapter. Classical integral representations of the Bernoulli and Euler numbers are given (p. 158).

With the exception of the chapters on hypergeometric series, the contents of Chapter 14 in the second Notebook have served as the basis of more research papers than any other topic of the Notebooks. It is quite curious, however, that many authors were unaware that their discoveries were hidden in the Notebooks. A few of the results in this chapter are found in Ramanujan's letters to Hardy [40, p. xxvi]. The chapter chiefly concerns relations between certain types of series. One of the formulas of the chapter that has been thoroughly scrutinized in recent times is Ramanujan's famous formula for $\zeta(2n+1)$. If α and β are positive numbers with $\alpha\beta=\pi^2$, and if n is any natural number, then (p. 177)

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^\infty \frac{k^{-2n-1}}{e^{2\alpha k}-1}\right\}=(-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^\infty \frac{k^{-2n-1}}{e^{2\beta k}-1}\right\}$$
$$-2^{2n}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k.$$

As is well known, the arithmetical nature of $\zeta(2n+1)$ is unknown. The above formula for $\zeta(2n+1)$ is probably the most interesting known formula for $\zeta(2n+1)$. For references to the many proofs of Ramanujan's formula, see [6] or [7] by the author. We cite two more examples from the wealth of fascinating formulae in this chapter. If $\alpha\beta=\pi$, then (p. 169)

$$\frac{\alpha}{4}\coth(n\alpha)-\frac{\beta}{4}\cot(n\beta)=\frac{n}{2}+\alpha\sum_{k=1}^\infty\frac{\sinh(2kn\alpha)}{e^{2k\alpha^2}-1}+\beta\sum_{k=1}^\infty\frac{\sin(2kn\beta)}{e^{2k\beta^2}-1}.$$

(No conditions on n are given, but the formula only holds if $0<\beta n<\pi$.) With no conditions listed for x and y (p. 175),

$$\frac{\pi}{4}\tan\left(\frac{\pi x}{2}\right)\tanh\left(\frac{\pi y}{2}\right)=y^2\sum_{k=1}^\infty\frac{\tanh\{(2k-1)\pi x/2y\}}{(2k-1)\{(2k-1)^2+y^2\}}+x^2\sum_{k=1}^\infty\frac{\tanh\{(2k-1)\pi y/2x\}}{(2k-1)\{(2k-1)^2-x^2\}}.$$

Series of the type

$$\sum_{k=1}^\infty\frac{\coth(k\pi)}{k^{4n-1}}\quad\text{and}\quad\sum_{k=0}^\infty\frac{(-1)^k\operatorname{sech}\{(2k+1)\pi/2\}}{(2k+1)^{4n+1}},$$

where n is a positive integer, are summed in closed form. A good reference for sums of this type is a paper of Nanjundiah [**27**]. Another interesting result is (p. 168)

$$\sum_{k=1}^{\infty} \frac{\sin^{2n+1} kx}{k} = \frac{\sqrt{\pi}}{2} \frac{\lfloor n-1/2}{\lfloor n},$$

where n is a natural number and $0 < x < \pi/(n+1)$. For proofs of several of the results of this chapter and references to other proofs, see [**7**].

Chapter 15 of the second Notebook is concerned with at least three apparently unrelated topics. There are some formulas which resemble those of the previous chapter. Ramanujan defines several concepts in the continuation of his unusual theory of series. Formulae from the theory of elliptic functions are also given.

Two related topics are developed in the following chapter. The first four pages develop the theory of the "partition" function

$$\prod(a,x) = (1+a)(1+ax)(1+ax^2)(1+ax^3)\cdots.$$

$\prod(a,x)$ also has the series representation

$$\sum_{n=0}^{\infty} \frac{a^n x^{n(n-1)/2}}{(1-x)(1-x^2)\cdots(1-x^n)}$$

and is called a basic hypergeometric function. On the chapter's opening page, we find classical results due to Cauchy, Heine and several others. Many of the theorems on $\Pi(a,x)$ can be found in Bailey's tract [**4**]. On page 196 of the second volume, there appears, in the words of Hardy [**21**, p. 222], "a remarkable formula with many parameters" involving several basic hypergeometric functions. This important formula contains the famous Jacobi triple product identity as a special case. For an especially elementary proof of Ramanujan's result and references to other proofs, see a paper of Andrews and Askey [**2**].

The second topic of Chapter 16 is a thorough development of the theory of

$$f(a,b) = 1 + (a+b) + ab(a^2+b^2) + (ab)^3(a^3+b^3) + (ab)^6(a^4+b^4) + \cdots,$$

which is actually one of the standard theta-functions. Ramanujan's work in this area of elliptic functions is well described in Chapter 12 of Hardy's book [**21**].

In Chapter 17 of the second Notebook, and in Chapter 15 of the first, Ramanujan develops the theory of

$$\exp\left[-\pi \frac{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2(1-x)^2+\&\text{c}}{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2+\&\text{c}}\right],$$

which, in contemporary notation, is equal to

$$\exp\left(-\pi \frac{F(1/2,1/2;1;1-x)}{F(1/2,1/2;1;x)}\right),$$

where $F(a,b;c;x)$ denotes the ordinary hypergeometric function. Some beautiful relations between integrals are found (vol. II, pp. 207–209). For example, if $\tan\alpha/\tan\beta = \sqrt{1+x}$, then

$$\int_0^{\alpha} \frac{d\varphi}{\sqrt{1+x\cos 2\varphi}} = \int_0^{\beta} \frac{d\varphi}{\sqrt{1-x^2\sin^4\varphi}}.$$

Most of the chapter lies in the domain of elliptic functions.

CHAPTER XVIII

1. $1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1 \cdot 1}{2 \cdot 4}\right)^2 x^2 + \left(\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^2 x^3 + \left(\frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}\right)^2 x^4 + \&c$

$= z(1-x) + \int z\,dx = \frac{2}{3}(1+x) + \frac{2}{3z}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$

2. $1 - \frac{1}{2^2}x - \frac{1^2 \cdot 3}{2^2 \cdot 4^2}x^2 - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2}x^3 - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2}x^4 - \&c$

$= z(1-x) + \frac{1}{2}\int z\,dx = \frac{2}{3}(2-x) + \frac{1}{3z}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$

3. The perimeter of an ellipse whose eccentricity is h, is

$2a\pi\left\{1 - \frac{1}{2^2}h^2 - \frac{1^2 \cdot 3}{2^2 \cdot 4^2}h^4 - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2}h^6 - \&c\right\}$

$= \pi(a+b)\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{a-b}{a+b}\right)^2 + \left(\frac{1 \cdot 1}{2 \cdot 4}\right)^2\left(\frac{a-b}{a+b}\right)^4 + \left(\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^2\left(\frac{a-b}{a+b}\right)^6 + \&c\right\}$

$= \pi\left\{3(a+b) - \sqrt{(a+3b)(3a+b)}\right\}$ nearly

$= \pi(a+b)\left\{1 + \frac{3x}{10 + \sqrt{4-3x}}\right\}$ very nearly where $x = \left(\frac{a-b}{a+b}\right)^2$.

N B. i. $\pi = 3\,14159\,26535,\,89793\,23846,\,26434.$

ii. $\log 10 = 2.30258\,50929,\,94045\,68401\,8.$

iii. $e^{-\pi} = .04321\,39182,\,63772\,25.$

iv. $e^{\frac{\pi}{2}} = 4.81047\,73809,\,65351\,65547\,3$

Cr. $\pi = \frac{355}{113}\left(1 - \frac{.0003}{3533}\right)$ very nearly.

$= \sqrt[4]{97\frac{1}{2} - \frac{1}{11}}$ nearly

4. $\frac{\sqrt{x}}{2}\left\{1 + \left(\frac{1}{2}\right)^2\frac{x}{3} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2\frac{x^2}{5} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2\frac{x^3}{7} + \&c\right\}$

$= \log\frac{1+e^{-y/2}}{1-e^{-y/2}} - 3\log\frac{1+e^{-3y/2}}{1-e^{-3y/2}} + 5\log\frac{1+e^{-5y/2}}{1-e^{-5y/2}} - \&c$

5. $\log\frac{16}{x} - \left(\frac{1}{2}\right)^2\frac{x}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2\frac{x^2}{2} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2\frac{x^3}{3} - \&c$

$= y - 4\left\{\log(1-e^{-y}) - 3\log(1-e^{-3y}) + 5\log(1-e^{-5y}) - \&c\right\}$

Approximations to π.

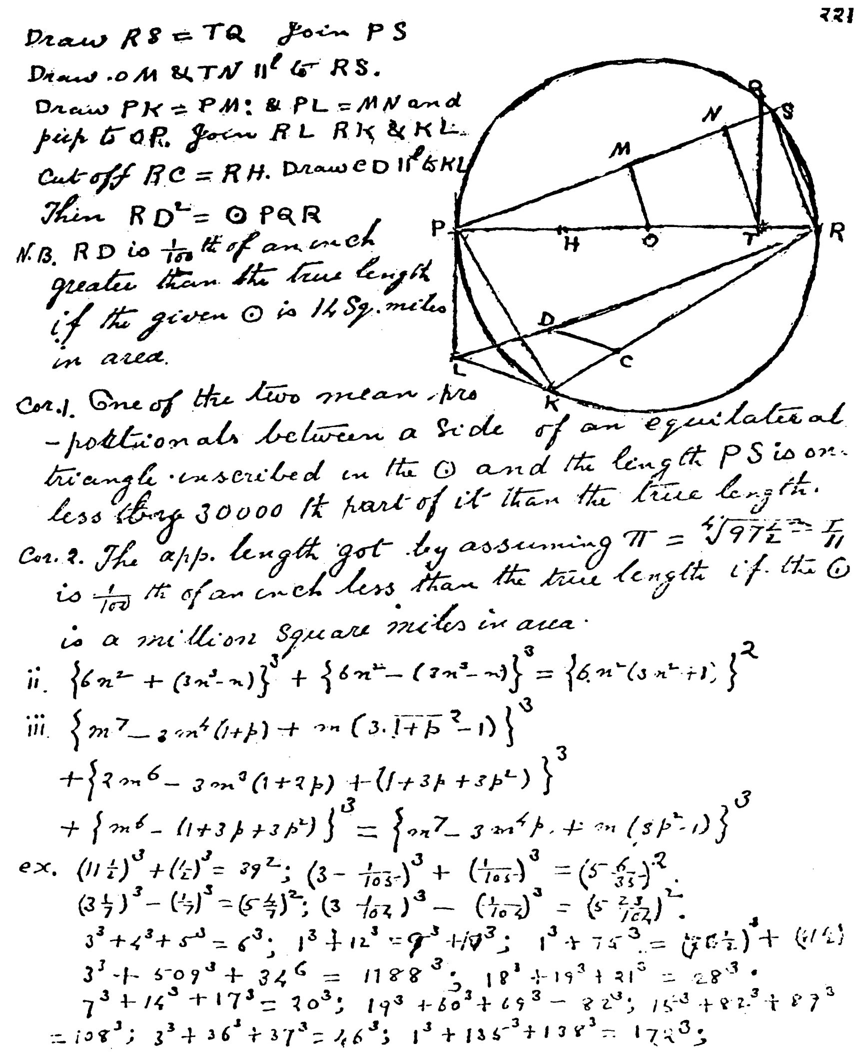
According to Ramanujan (vol. II, Chapter 18, p. 217),

$$\pi = \frac{355}{113}\left(1 - \frac{.0003}{3533}\right) \quad \text{very nearly, and} \quad \pi = \sqrt[4]{97\tfrac{1}{2} - \tfrac{1}{11}} \quad \text{nearly.}$$

In this instance, "very nearly" means that the approximation is good to 15 decimal places, and "nearly" means that the approximation is valid to 9 decimal places. These approximations to π, as well as other curious approximations, can be found in Ramanujan's paper [**34**], [**40**, pp. 23–39]. The latter approximation to π given above is used later in the chapter (p. 225) where Ramanujan indicates a method for almost squaring the circle. See also a one page paper [**33**], [**40**, p. 22] that Ramanujan wrote on squaring the circle.

Several pages of Chapter 18 are devoted to series involving trigonometric and hyperbolic trigonometric functions. A typical example is (p. 218)

$$\sum_{n=0}^{\infty}(-1)^n\frac{\cos\{(2n+1)\theta\} + 2\cos\{\frac{1}{2}(2n+1)\theta\}\cosh\{\frac{1}{2}(2n+1)\sqrt{3}\,\theta\}}{(2n+1)\cosh\{\frac{1}{2}(2n+1)\sqrt{3}\,\pi\}} = \frac{\pi}{8}.$$


221

Draw RS = TQ Join PS
Draw OM & TN ‖l to RS.
Draw PK = PM; & PL = MN and perp to OP. Join RL RK & KL.
Cut off RC = RH. Draw CD ‖l to KL
Then $RD^2 = \odot PQR$
N.B. RD is $\frac{1}{100}$ th of an inch greater than the true length if the given ⊙ is 14 Sq. miles in area.
Cor. 1. One of the two mean proportionals between a side of an equilateral triangle inscribed in the ⊙ and the length PS is one less [illegible] 30000 th part of it than the true length.
Cor. 2. The app. length got by assuming $\pi = \sqrt[4]{97\frac{1}{2} - \frac{1}{11}}$ is $\frac{1}{100}$ th of an inch less than the true length if the ⊙ is a million square miles in area.

ii. $\{6n^2 + (3n^2-n)\}^3 + \{6n^2 - (3n^2-n)\}^3 = \{6n^2(3n^2+1)\}^2$

iii. $\{m^7 - 3m^4(1+p) + m(3\overline{1+p}^2 - 1)\}^3$
$+\{2m^6 - 3m^3(1+2p) + (1+3p+3p^2)\}^3$
$+\{m^6 - (1+3p+3p^2)\}^3 = \{m^7 - 3m^4p + m(3p^2-1)\}^3$

ex. $(11\frac{1}{2})^3 + (\frac{1}{2})^3 = 39^2$; $(3 - \frac{1}{105})^3 + (\frac{1}{105})^3 = (5 - \frac{6}{35})^2$.
$(3\frac{1}{7})^3 - (\frac{1}{7})^3 = (5\frac{4}{7})^2$; $(3 - \frac{1}{102})^3 - (\frac{1}{102})^3 = (5 - \frac{25}{102})^2$.
$3^3 + 4^3 + 5^3 = 6^3$; $1^3 + 12^3 = 9^3 + 10^3$; $1^3 + 75^3 = $ [illegible]
$3^3 + 509^3 + 34^6 = 1188^3$; $18^3 + 19^3 + 21^3 = 28^3$.
$7^3 + 14^3 + 17^3 = 20^3$; $19^3 + 60^3 + 69^3 = 82^3$; $15^3 + 82^3 + 87^3$
$= 108^3$; $3^3 + 36^3 + 37^3 = 46^3$; $1^3 + 135^3 + 138^3 = 172^3$;

The source of the taxicab anecdote.

A simpler example (p. 219)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^7 \cosh\left\{\frac{1}{2}(2n+1)\sqrt{3}\,\pi\right\}} = \frac{\pi^7}{23{,}040}$$

was established by Watson [**48**].

Most readers are undoubtedly familiar with Hardy's famous story about the taxicab number [**40**, p. xxxv], [**21**, p. 12]. On the way to visiting Ramanujan, when he was seriously ill, Hardy rode in a taxicab numbered 1729. Hardy expressed his feelings to Ramanujan that the number is rather a dull one. But Ramanujan exclaimed that 1729 is, indeed, very interesting, for 1729 is the first natural number that can be written as the sum of two cubes in two distinct ways. On the bottom of page 225 in the second Notebook, we find these representations

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

Ramanujan also gives other examples of this sort, like

$$19^3 + 60^3 + 69^3 = 82^3 \quad \text{and} \quad 133^3 + 174^3 = 45^3 + 196^3.$$

Chapters 19–21 in the second volume concern elliptic functions and are a continuation of the theory developed in Chapters 16 and 17. Several pages consist solely of very complicated expressions involving radicals. These are computations involving singular moduli. The material is related to Ramanujan's work in [**34**], [**40**, pp. 23–39]. Watson has made a thorough study of that material in Chapters 16–21 which is connected with elliptic functions. See the several papers of Watson listed in the bibliography of [**21**], as well as Chapter 12 there.

The bulk of the miscellaneous material after Chapter 16 in the first Notebook is related to elliptic functions. On page 274 of the second volume, the germ for Ramanujan's paper [**36**], [**40**, pp. 59–67] is found in that

$$\int_0^\infty \frac{\cos nx}{e^{2\pi\sqrt{x}}-1}\,dx$$

is evaluated for certain values of n. See also [**32**], [**40**, p. 327] and a paper of Watson [**50**]. The miscellaneous notes also contain additional integral theorems of the Fourier type, some work on functional equations, several additional continued fraction expansions, definitions of legitimate and illegitimate convergent and divergent series (p. 348), and more theorems related to Chapters 14 and 16–21.

Some of Ramanujan's work on number theory is found in the miscellaneous material. Ramanujan, no doubt, did not have proofs of many of his statements in the theory of numbers. In fact, some of his claims are false. But his insights and intuition are remarkable. On page 307 (volume II), Ramanujan gives a form of the prime number theorem by stating that "The no. of prime nos. between A and B

$$=\int_A^B \frac{dx}{\log x} \qquad \text{nearly."}$$

(Other estimates for the number of primes can be found on pages 317–318.) He also says that the number of integers between A and B that can be written as a sum of two squares is

$$C\int_A^B \frac{dx}{\sqrt{\log x}} \qquad \text{nearly,}$$

where $C=.764\ldots$. This statement is asymptotically true, and a proof of this was first given by Landau [**21**, pp. 62–63]. The result was also communicated to Hardy in one of Ramanujan's letters [**40**, p. xxiv]. Hardy discusses this statement at length [**21**, pp. 8–9, 19, 62–63], and apparently there is some controversy about how good of an approximation Ramanujan thought that he had. For a full discussion of this and references to recent work, see a paper of Shanks [**45**]. Ramanujan gives representations of primes as sums or differences of squares, but most of these results were discovered long ago by Fermat.

The third Notebook is somewhat of a continuation of the heterogeneous material at the end of the second Notebook. More material on sums of squares and divisors of integers is found. On page 371, Ramanujan gives some long intervals of composite numbers, for example, 370,261 to 370,373 and 2,010,733 to 2,010,881. As usual, let $\pi(x)$ denote the number of primes not exceeding x. Ramanujan calculated $\pi(x)$ for several values of x. For example, Ramanujan claims that $\pi(10^8)=5{,}761{,}460$; the correct value is 5,761,455. The seeds for Ramanujan's paper on highly composite numbers [**37**], [**40**, pp. 78–128] are found on page 372.

On page 374, Ramanujan announces some expressions for certain values of the exponential function. For example, he states that

$$e^{\frac{\pi}{4}\sqrt{78}}=4\sqrt{3}\,(75+52\sqrt{2}\,) \tag{4}$$

and

$$e^{\frac{\pi}{4}\sqrt{130}}=12(323+40\sqrt{65}\,). \tag{5}$$

These and similar findings are found in Ramanujan's paper [**34**], [**40**, pp. 23–39], where he writes that these are "approximate formulae". Samuel Wagstaff has kindly computed the numbers in (4) and (5). The left and right hand sides of (4) and (5), respectively, are

$$1029.1091087457\ldots, \qquad 1029.1091087695\ldots$$

and

$$7745.88371918324\ldots, \qquad 7745.88371918330\ldots.$$

On page 375, we find that

$$e^{\pi\sqrt{58}} = 24591257751.99999982\ldots$$

is "almost" an integer. Examples like those in this paragraph can be explained by means of the theory of complex multiplication and elliptic functions. Moreover, instances of this sort are found in the works of Hermite and Kronecker [**46**, p. 357] several years earlier. A particular example of this type of approximation recently served as the basis of an "April Fool's" joke by Martin Gardner [**14**, p. 126] in *Scientific American.*

Examples of equalities between fourth and higher powers of integers are found on pages 384–386 with some general rules on how to construct such examples.

Half of page 390 and pages 391 and 392 are written upside down. Page 391 is somewhat enigmatic. As mentioned earlier, Ramanujan never indicated any knowledge of complex analysis. Phrases such as "It can be shewn, by the theory of residues, that..." [**38**], [**40**, p. 129] are thought to have been supplied by Hardy. On page 391, however, Ramanujan states several Mellin transforms, and the words "contour integration" appear. Furthermore, Ramanujan also uses the notation $\Gamma(n)$, where elsewhere in the Notebooks, Ramanujan would have used $\underline{|n-1}$ instead. Probably, this single page dates from Ramanujan's years at Cambridge.

Epilogue

In discussions about Ramanujan, the question "How really great a mathematician was he?" inevitably arises. In the domains of infinite series, elliptic functions, and continued fractions, very few in the history of mathematics have been his equal. Our judgement, however, is somewhat clouded by the singular nature of his "proofs". Hardy [**21**, p. 7] deemed that "It (Ramanujan's mathematics) has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange." Much of Ramanujan's work which did not seem inevitable at the time it was discovered has now become more inevitable as we see how it fits into the rest of mathematics. Thus, perhaps the hour of final judgement is not yet at hand. But of Ramanujan's love and devotion for mathematics, there can be no doubt. And perhaps he would have said with Shelley, in the latter's Hymn to Intellectual Beauty,

I vowed that I would dedicate my powers
To thee and thine; have I not kept the vow?

This paper has benefited from the comments of several mathematicians. In particular, the author wishes to thank George Andrews, Richard Askey, Shashi Gadia, Emil Grosswald, and Kenneth Stolarsky.

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