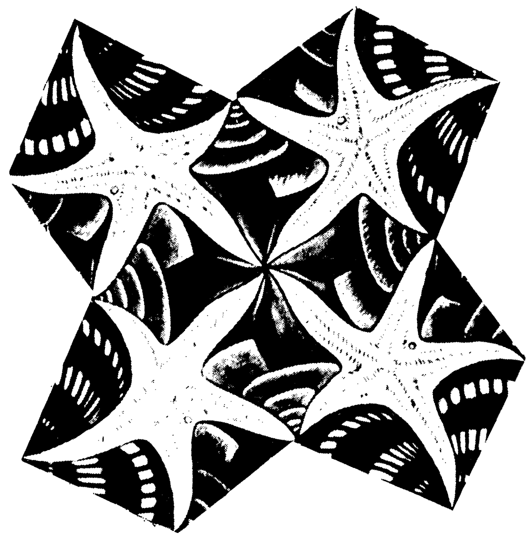
Tiling the Plane with Congruent Pentagons

A problem for anyone to contribute to: a survey of the growing but incomplete story of pentagonal tilings of the plane.

DORIS SCHATTSCHNEIDER
Moravian College,
Bethlehem, PA 18018

The importance of recreational mathematics and the involvement of amateur mathematicians has been dramatically demonstrated recently in connection with the problem of tiling the plane with congruent pentagons. The problem is to describe completely all pentagons whose congruent images will tile the plane (without overlaps or gaps). The problem was thought to have been solved by R. B. Kershner, who announced his results in 1968 **[18]**, **[19]**. In July, 1975, Kershner's article was the main topic of Martin Gardner's column, "Mathematical Games" in *Scientific American.* Inspired by the challenge of the problem, at least two readers attempted their own tilings of pentagons and each discovered pentagons missing from Kershner's list. New interest in the problem has been aroused and both amateur and professional mathematicians are presently working on its solution.

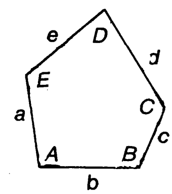


Tilings by convex polygons

A polygon is said **to tile** the plane if its congruent images cover the plane without gaps or overlaps. The pattern formed in this manner is called a **tiling** of the plane, and the congruent polygons are called its tiles. A **vertex of the tiling** is a point at which 3 or more tiles meet. It is well known that any triangle or any quadrilateral can tile the plane. In fact, any single triangle or quadrilateral can be used as a "generating" tile in a tiling of the plane which is **tile-transitive** (isohedral). This simply means that the generating tile can be mapped onto any other tile by an isometry of the tiling. The translations, rotations, reflections, and glide-reflections which map a tiling onto itself make up the **symmetry group** of the tiling. Thus, in terms of group theory, a tile-transitive tiling is one whose symmetry group acts transitively on the tiles.

Also, for an arbitrary triangle or quadrilateral there always exists a tiling which in addition is **edge-to-edge.** This means that for any two tiles, exactly one of the following holds: (i) they have no points in common, (ii) they have exactly one point in common, which is a vertex of each tile (such a point is also a vertex of the tiling), (iii) their intersection is an edge of each tile. Typical tilings of triangles and of quadrilaterals having these properties are shown in **[5]**, **[25]**.

1. In a given tiling, pentagons marked with a dot are oppositely congruent to those which are unmarked, i.e., the plain tiles are 'face up', and the marked tiles are 'face down'.
2. In each tiling there is outlined a minimal block of one or more pentagons which generates the tiling when acted on by the symmetry group of the tiling.
3. Angles A, B, C, D, E of one pentagonal tile are identified in each tiling for use with Tables I, II, III. Sides a, b, c, d, e of that tile correspond to the following labeling:

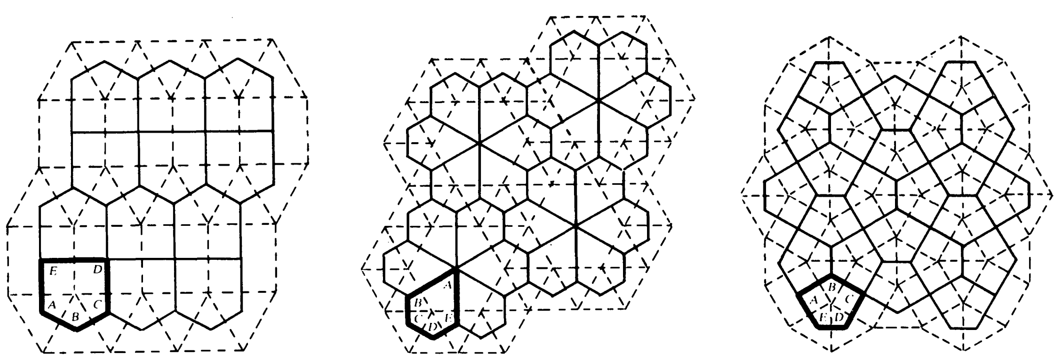


Key to all diagrams

If we ask the natural question, "Do convex polygons of five or more sides tile the plane?", it is clear that the obvious general answer is "Not always." It is clear that not every convex pentagon tiles the plane — a regular pentagon is a prime example. However, regular hexagons do tile the plane (how many times have you seen this pattern on a 1930's bathroom floor?), but not all hexagons tile. A complete description of all hexagons which do tile the plane was discovered independently by several mathematicians and is discussed in **[2]**, **[5]**, **[14]**, **[18]**, **[19]**, **[28]**. It can also be demonstrated that no convex polygon of more than six sides can tile the plane. Thus, the problem of describing all convex pentagons which tile the plane is the only unanswered part of our question. In what follows, we make several observations related to the pentagonal tiling problem and report on the most recent contributions to its solution.

Discovering tilings by pentagons

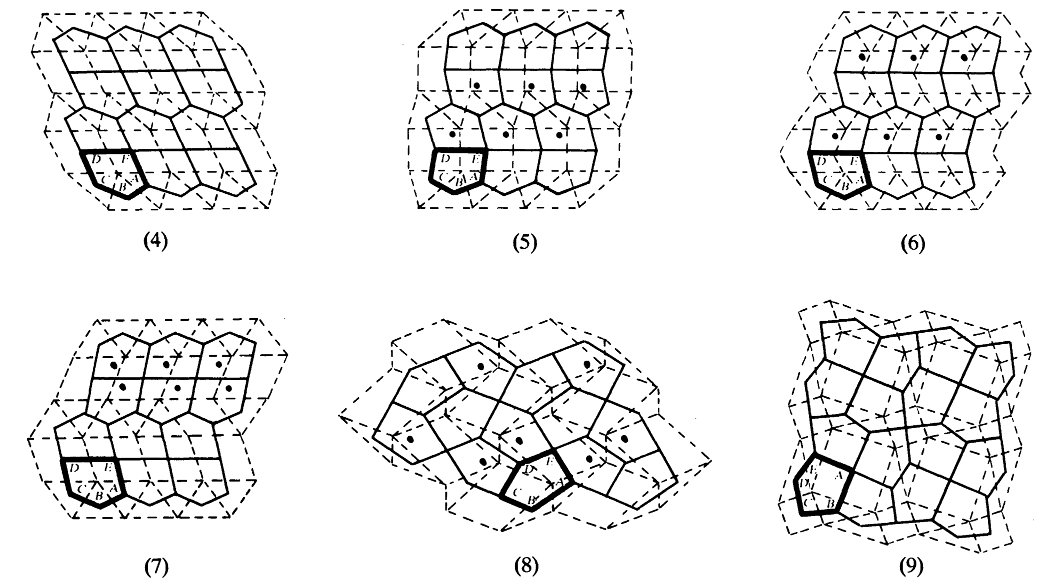
Three of the oldest known pentagonal tilings are shown in FIGURE 1. As Martin Gardner observed in **[5]**, they possess "unusual symmetry". This symmetry is no accident, for these three tilings are the duals of the only three Archimedean tilings whose vertices are of valence 5. The underlying Archimedean tilings are shown in dotted outline. Tiling (3) of FIGURE 1 has special aesthetic appeal. It is said to appear as street paving in Cairo; it is the cover illustration for Coxeter's *Regular Complex Polytopes*, and was a favorite pattern of the Dutch artist, M. C. Escher. Escher's sketchbooks reveal that this tiling is the unobtrusive geometric network which underlies his beautiful "shells and starfish" pattern. He also chose this pentagonal tiling as the bold network of a periodic design which appears as a fragment in his 700 cm. long print "Metamorphosis II."



(1) (2) (3)

The three pentagonal tilings which are duals of Archimedean tilings. The underlying Archimedean tiling is shown in dotted outline.

FIGURE 1.



Tile-transitive, edge-to-edge tilings by pentagons, obtained as duals of vertex-transitive tilings (shown in dotted outline). Only in the case of (6) is a non-convex tile necessary in the underlying tiling. Tiling (8) is discussed in [4].

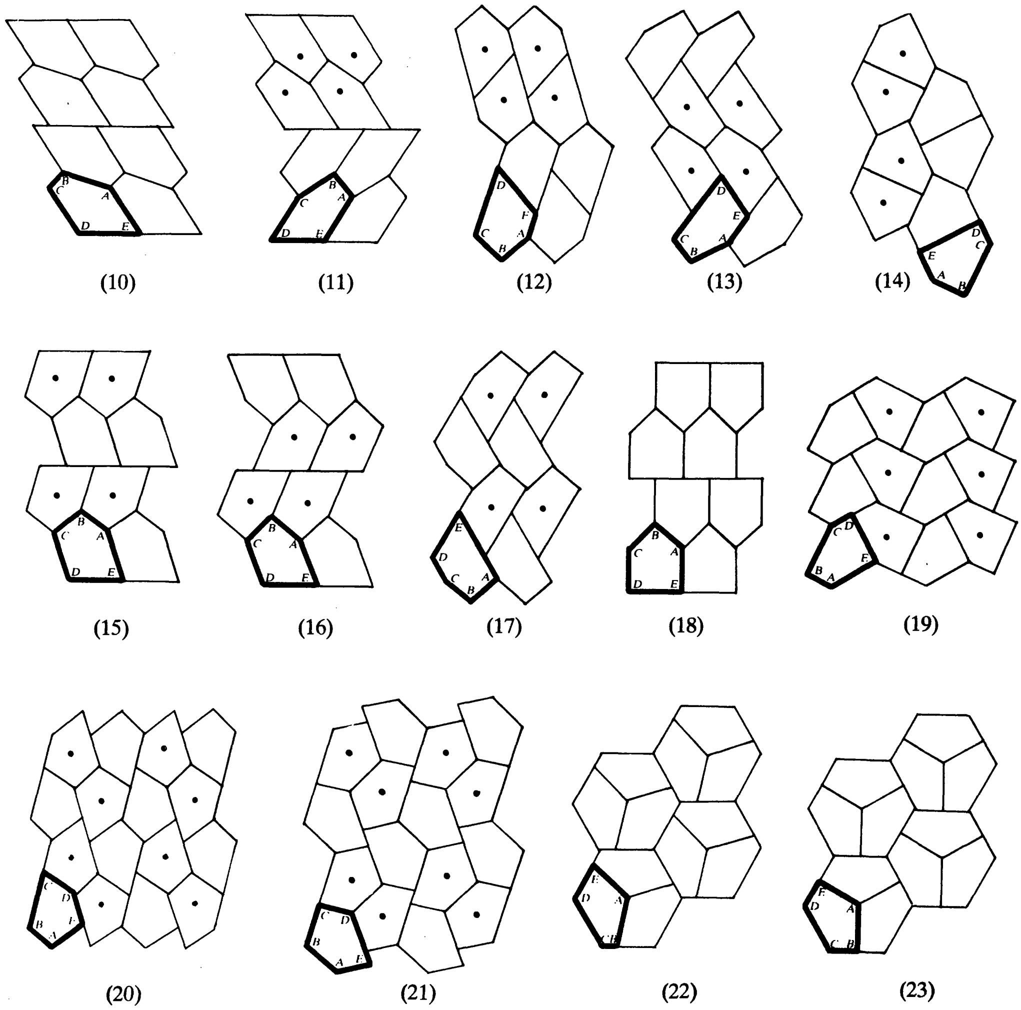
FIGURE 2.

Tiling (3) can also be obtained in several other ways. Perhaps most obviously it is a grid of pentagons which is formed when two hexagonal tilings are superimposed at right angles to each other. F. Haag noted that this tiling can also be obtained by joining points of tangency in a circle packing of the plane [**12**]. It can also be obtained by dissecting a square into four congruent quadrilaterals and then joining the dissected squares together [**26**]. The importance of these observations is that by generalizing these techniques, other pentagonal tilings can be discovered.

The three Archimedean tilings which have as duals the pentagonal tilings in FIGURE 1 are vertex transitive tilings. An edge-to-edge tiling by polygons is called **vertex transitive** (isogonal) if the symmetry group of the tiling is transitive on the vertices of the tiling. FIGURE 2 shows six other edge-to-edge pentagonal tilings that arise as duals of vertex-transitive tilings of the plane. Recently, B. Grünbaum and G. C. Shephard showed that the nine tilings of FIGURES 1 and 2 are the only distinct "types" of pentagonal tilings which are edge-to-edge and tile-transitive. Roughly speaking, two "types" of tilings will differ if they have different symmetry groups or if the relationship of tiles to their adjacent tiles differs. Details are given in [**6**]. In addition, these mathematicians have classified all vertex-transitive tilings, and their list shows that no such tiling by convex pentagons is possible [**7**].

If we begin with a tiling by congruent convex hexagons, then pentagonal tilings can arise in two different ways. First, it may be possible to superimpose the hexagonal tiling on itself so as to produce a tiling by congruent pentagons. Tilings (8) and (9) of FIGURE 2 can arise in this way. Also, beginning with a hexagonal tiling, it may be possible to dissect each hexagon into two or more congruent pentagons, thus producing a pentagonal tiling. Many tilings of FIGURES 1, 2, and 3 can be viewed in this manner. Three other examples of such tilings given in FIGURE 4 also serve to illustrate other properties of pentagonal tilings that can occur. Note that (24) is tile-transitive but not edge-to-edge, tiling (25) is edge-to-edge but not tile-transitive, and tiling (26) is neither edge-to-edge nor tile-transitive.

Finally, experimentation in fitting pieces together or adding or removing lines from other geometric tilings can lead to the discovery of pentagonal tilings. FIGURE 5 illustrates this with two tilings by a simple "house" shape pentagon.



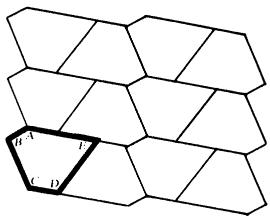
Tile-transitive tilings by pentagons which are not edge-to-edge.

FIGURE 3.

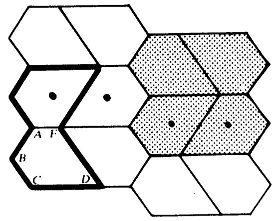
Methodical attacks

Pentagonal tilings appear as illustrations in several early papers which explore the general problem of classifying plane repeating patterns (especially **[11]**), but it appears that the first methodical attack on classifying pentagons which tile the plane was done in 1918 by K. Reinhardt in his doctoral dissertation at the University of Frankfurt **[28]**. He discovered five distinct types of pentagons, each of which tile the plane. More precisely, he stated five different sets of conditions on angles and sides of a pentagon such that each set of conditions is sufficient to ensure that (i) a pentagon fulfilling these conditions exists, and (ii) at least one tiling of the plane by such a pentagon exists. Each of these five sets of conditions defines a *type* of pentagon; pentagons are considered to be of different types only if they do not satisfy the same set of conditions. Many distinct tilings can exist for pentagons of a given type. Reinhardt no doubt hoped that his five types constituted a complete solution to the problem, but he was unable to show that a tiling pentagon was necessarily one of these types.

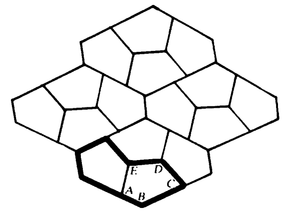
Each of the five types described by Reinhardt (called types 1 — 5 in TABLE I) can generate a tile-transitive tiling of the plane. His thesis completely settled the problem of describing all convex



(24)



(25)



(26)

Tilings by pentagons obtained by dissecting hexagonal tilings. Tiling (24) is tile-transitive, but in (25) and (26), it is impossible to map one pentagon in an outlined block onto the other pentagon in that block by a symmetry of the tiling. The shaded portion of (25) shows a "double hexagon" which has been dissected into 4 congruent pentagons.

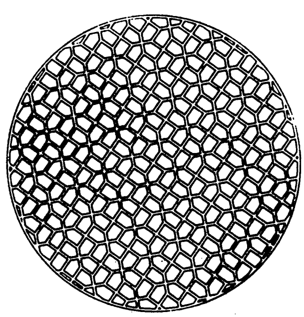
FIGURE 4.

hexagons which tile the plane — there are just three types, and each of these can generate a tile-transitive tiling. The fact that if a hexagon can tile the plane at all, then that same hexagon can generate a tile-transitive tiling, considerably simplifies the hexagonal tiling problem. Unfortunately, this result is not true for pentagons and this may be the reason that Reinhardt did not pursue the problem further by trying to find other types of pentagons which tile.

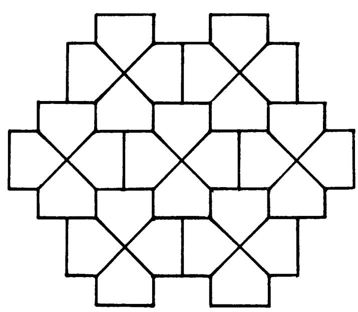
In [**14**], pp. 81–91, Heesch and Kienzle methodically explore the problem of describing types of pentagons which can generate tile-transitive tilings and affirm that Reinhardt's five types are the only convex ones possible. Most recently, B. Grünbaum and G. C. Shephard, using a classification scheme for tilings, found exactly 81 "types" of tile-transitive tilings of the plane by quite general "tiles" [**6**], and, as a result, have not only confirmed these earlier results, but also have shown that there are exactly twenty-four distinct "types" of tile-transitive tilings by pentagons according to their classification scheme [**9**]. Nine of these are edge-to-edge (tilings (1)–(9)); the other fifteen are illustrated by tilings (10)–(24). TABLE I summarizes information on these tilings.

In 1968, R. B. Kershner of Johns Hopkins University announced that there are 8 types of pentagons which tile the plane [**18**], [**19**]. He devised a method different from Reinhardt's for classifying pentagons which can tile, and this scheme left out any assumptions of tile transitivity for associated tilings. Happily, his search yielded three classes of pentagons not on Reinhardt's list. His 3 new pentagonal tiles (types 6, 7, 8 on TABLE II) each have an associated tiling which is edge-to-edge and not tile-transitive (FIGURE 6).

Although Kershner's claim — that these three additional types of pentagons completed the list of



(27)



(28)

A "house shape" pentagon tiles in a variety of ways. Tiling (27) is a Chinese lattice design (*Chinese Lattice Designs*, D. S. Dye, Dover, 1974, p. 340); (28) is a familiar geometric pattern of interlocked St. Andrews crosses. Other patterns are (1), (14), (18).

FIGURE 5.

Type, with characterizing conditions	Tile-Transitive tilings	Additional Conditions necessary for tiling	International Notation for Symmetry Group of Tiling (see [30])	Isohedral type (from [6])	Type from [9]	Other tilings shown
1. $D+E=\pi$	(24)		$p2$	$IH4$	P_5-4	
	(10)	$a=d$	$p2$	$IH4$	P_5-5	
	(11)	$a=d$	pgg	$IH5$	P_5-8	
	(4)	$a=d$	$p2$	$IH23$	P_5-18	
	(5)	$a=d$	pmg	$IH24$	P_5-19	
	(12)	$b=c$	pgg	$IH5$	P_5-6	(25); (33) if $a=e, b=d$, $D=E=\pi/2$
	(13)	$a+e=d$	pgg	$IH5$	P_5-7	
	(14)	$a+d=c$	pgg	$IH6$	P_5-12	
	(15)	$a=d, b=c$	pg	$IH2$	P_5-1	(35) if $A+2D=2\pi$, $a=b=c=d$
	(16)	$a=d, b=c$	pgg	$IH5$	P_5-9	
	(6)	$a=d, b=c$	cm	$IH22$	P_5-17	(40), (41) if $D=80°$ and $a=b=c=d=e$
	(7)	$a=d, b=c$	pgg	$IH25$	P_5-20	
	(17)	$b=c, a=d+e$	pg	$IH2$	P_5-2	
	(18)	$D=E=\pi/2, A=C$, $a=d, b=c$	pmg	$IH15$	P_5-14	
	(1)	$D=E=\pi/2, A=C$, $a=d, b=c$	cmm	$IH26$	P_5-21	(27), (28) if $B=\pi/2$
	(19)	$D+B=\pi, c=e$, $a=b+d$	pg	$IH3$	P_5-3	
2. $C+E=\pi$	(20)		pgg	$IH6$	P_5-10	(26) if $A+C=\pi, d=e$
$a=d$	(21)		pgg	$IH6$	P_5-11	
	(8)	$c=e$	pgg	$IH27$	P_5-22	(36) if $D+2E=2\pi$, $a=c=d=e$
3. $A=C=D=2\pi/3$	(22)		$p3$	$IH7$	P_5-13	
$a=b$	(23)	$B=E=\pi/2$	$p31m$	$IH16$	P_5-15	
$d=c+e$						
4. $A=C=\pi/2$	(9)		$p4$	$IH28$	P_5-23	(27) if $E=\pi/2$ and $a=b=e$
$a=b$, $c=d$	(3)	$D=E$	$p4g$	$IH29$	P_5-24	
5. $A=\pi/3$, $C=2\pi/3$, $a=b, c=d$	(2)		$p6$	$IH21$	P_5-16	

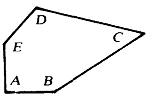
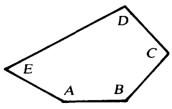
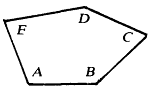
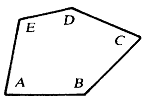
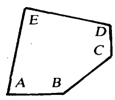
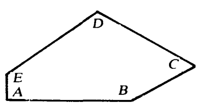
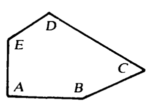
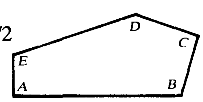
THE FIVE TYPES OF PENTAGONS which can generate tile-transitive tilings of the plane. The tiles for a given transitive tiling are not always uniquely of one type (e.g. the tiles of tiling (19) satisfy conditions on angles and sides of both types 1 and 2). In this table we have listed each transitive tiling only once, thus identifying its tiles by just one type.

TABLE I

pentagons which tile — was later shown false, still his discovery was important. It confirmed that there are pentagons whose associated tilings cannot be tile-transitive, thereby answering a question raised by J. Milnor in [**24**, p. 499]. His search for pentagons which tile had been a methodical one; yet still, in his own words, he "made at least 2 errors, one of commission, and one of omission".

Contributions by amateurs

The publication of the problem and Kershner's list by Martin Gardner in [**5**] stimulated amateurs to try to find pentagons which tile. Richard James III, a California computer scientist, read the

Type, with characterizing conditions	Illustrative tile and tiling number		Symmetry group of tiling	Remarks
6. $C+E=\pi$ $A=2C$ $a=b=e$ $c=d$		(29)	$p2$	2-block transitive. Associated block tiling (29)–B is isohedral type $IH4$.
7. $2B+C=2\pi$ $2D+A=2\pi$ $a=b=c=d$		(30)	pgg	2-block transitive. Associated block tiling (30)–B is isohedral type $IH6$.
8. $2A+B=2\pi$ $2D+C=2\pi$ $a=b=c=d$		(31)	pgg	2-block transitive. Associated block tilings (31)–B and (31)–B' are type $IH6$.
9. $2E+B=2\pi$ $2D+C=2\pi$ $a=b=c=d$		(34)	pgg	2-block transitive. Associated block tiling (34)–B is type $IH53$.
10. $E=\pi/2$ $A=\pi-D$ $B=\frac{\pi+D}{2}$ $C=\frac{\pi-D}{2}$ $a=e=b+d$		(32) (33) if $D=\pi/2$, $b=d$	$p2$ cmm	D is bounded: $\pi-\tan^{-1}(4/3)<D<\tan^{-1}(4/3)$ 3-block transitive. Associated block tilings (32)–B and (33)–B are type $IH4$.
11. $A=\frac{\pi}{2}$ $C+E=\pi$ $2B+C=2\pi$ $d=e=2a+c$		(37)	pgg	2-block transitive. Associated block tiling is type $IH6$.
12. $A=\frac{\pi}{2}$ $C+E=\pi$ $2B+C=2\pi$ $e+c=d=2a$		(38)	pgg	2-block transitive. Associated block tiling is type $IH6$.
13. $A=C=\pi/2$ $B=E=\pi-D/2$ $c=d$ $2c=e$		(39)	pgg	2-block transitive. Associated block tiling is IH5.

Eight Types of Pentagons which Tile, but for Which No Tile-Transitive Tiling Exists.

TABLE II.

problem and decided not to look at Kershner's list, but see if he could find some pentagonal tilings himself. Familiar with the common tiling by octagons and squares, and noting that an octagon is easily dissected into four congruent pentagons by perpendicular lines through its center, he attempted to change the familiar tiling into a pentagonal one. He was successful and sent an example to Martin Gardner [**17**]. FIGURE 7 shows two tilings by pentagons of James's type (type 10 on TABLE II). James's discovery served to point out a hidden assumption in Kershner's search — he had, in fact, only been looking for pentagons which could tile in an edge-to-edge manner, or in a manner in which every tile was surrounded by six vertices of the tiling (as, for example, in tilings (10) through (24)). James's pentagons are only capable of tiling in a manner which is not tile-transitive and not edge-to-edge. In addition, some pentagons in this tiling are surrounded by 5 vertices of the tiling, others are surrounded by 7 vertices of the tiling.

Marjorie Rice, a Californian with no mathematical training beyond "the bare minimum they required ... in high school over 35 years ago", also read Gardner's column and began her own methodical attack on the problem. Her approach was to consider the different ways in which the vertices of a single pentagon could "come together" to form a vertex of a tiling by congruent images of that pentagon. These considerations forced conditions on the angles and sides of the pentagon if it was to tile, thus giving either a description of a pentagon which could tile in a prescribed manner, or forcing the conclusion that no pentagon could be constructed which satisfied the conditions.

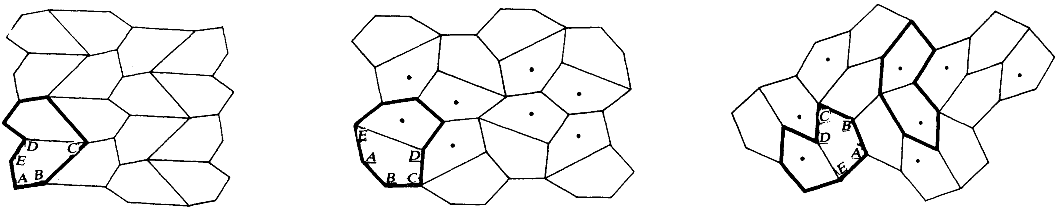
This essentially combinatorial search yielded over forty different tilings by pentagons and included a tiling by a new type of tile not on Kershner's list. Her discovery (type 9 on TABLE II) showed that Kershner's search (which was similar to hers) erroneously eliminated the possiblity of this type of edge-to-edge tiling. A later methodical search by Rice considered twelve different classes of pentagons, each class corresponding to a description of which sides of a given pentagon are equal. Possible tilings for each class were sought — and for every class at least one tiling was found. Over 58 diagrams of distinct tilings were produced in this effort, most of them non-transitive tilings by tiles of type 1. Even though she missed several of the 24 tile-transitive tilings, her scheme was complete enough to produce a tiling for every one of the pentagons of types 1–10 in TABLES I and II. No other new types of tiles were produced in this effort. FIGURE 8 shows three of Rice's tilings for the class of pentagons having four equal sides, including tiling (34) associated to her type 9. (The pentagons in tilings (30) and (31), FIGURE 6, also have 4 equal sides.)

Block transitive tilings

The solution of the hexagonal tiling problem was simplified by a theorem which reduced the problem to one of hexagons capable of producing tile-transitive tilings. Although this theorem is clearly false for the case of pentagons which tile, we can observe that for each pentagon of types 1 through 10, there exists a tiling containing a minimal 'block' of congruent pentagons which has the property that (i) the tiling consists of congruent images of this block and (ii) this block can be mapped onto any other congruent block by an isometry of the tiling. If a minimal such block contains n pentagons, we will say that the tiling is ***n*-block transitive.** Thus a tile-transitive tiling is 1-block transitive. We remark that for $n \geqq 2$, a given n-block transitive tiling may have several non-congruent minimal n-blocks. We have outlined two such 2-blocks in tiling (31).

If we remove the interior edges of pentagons in the heavily outlined blocks shown in the tilings in FIGURES 6, 7, and 8, the resulting transitive block tilings reveal information not immediately apparent from these pentagonal tilings alone. In FIGURE 9, we can see that each of Kershner's tilings (29), (30), (31) produces a block tiling in which each block is surrounded by six vertices of the tiling. Thus, these block tilings are formed by non-convex tiles which are topological hexagons. The Kershner tilings are obtained from these 'hexagonal' tilings by bisecting each 'hexagon' into two congruent pentagons. This method parallels the technique noted earlier of obtaining pentagonal tilings by bisecting hexagonal tilings. The reader can verify that the 2-block transitive tilings (25), (26), (35) are also obtained by bisecting blocks which tile as topological hexagons.

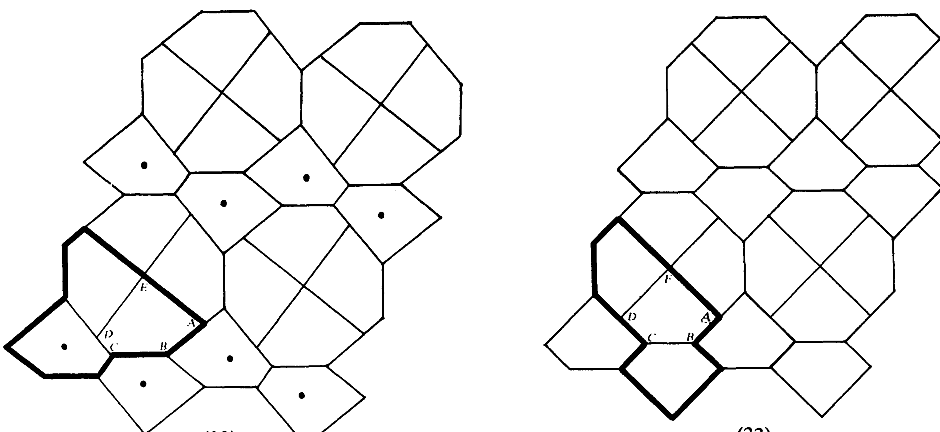
Tilings (32), (33) of James's pentagons also have associated block tilings of topological hexagons (FIGURE 9). In this case, however, each 'hexagon' has been dissected into 3 congruent pentagons, an occurrence that has no parallel in any of the pentagonal tilings known prior to James's discovery. It might be appropriate to note here that the symmetry group of the block tiling (33) – B of FIGURE 9 is $p2$, which is a proper subgroup of the symmetry group of the pentagonal tiling (33). (The names of tiling symmetry groups are given in TABLES I and II.) This occurrence is not surprising, since the removal of some edges of the pentagonal tiling can cause loss of some symmetries of the pattern (in this case, reflections).



(29) (30) (31)

Tilings for each of the three types of pentagons discovered by R. B. Kershner. Each tiling is 2-block transitive, and there exists no tile transitive tiling for these types of pentagons. Two non-congruent 2-blocks are outlined in tiling (31); each of these generates a block-transitive tiling.

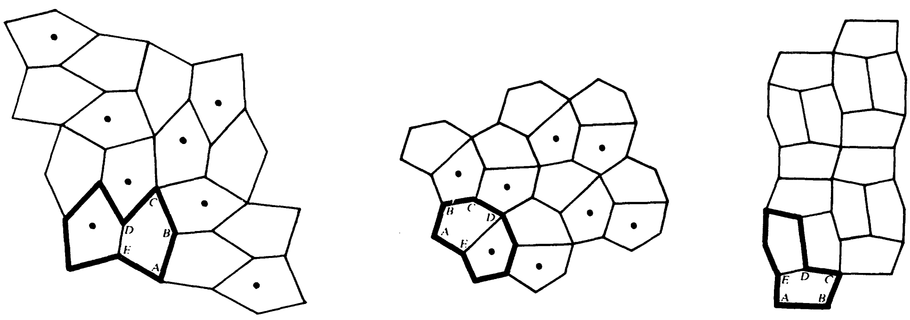
FIGURE 6.



(32) (33)

Tilings by type of pentagon discovered by Richard James III. The tilings consist of strips of attached octagons, separated by strips of bow ties. Each octagon contains four pentagons, and each bow tie contains two pentagons, which are in opposite orientation from those in the octagons. The tilings are 3-block transitive.

FIGURE 7.



(34) (35) (36)

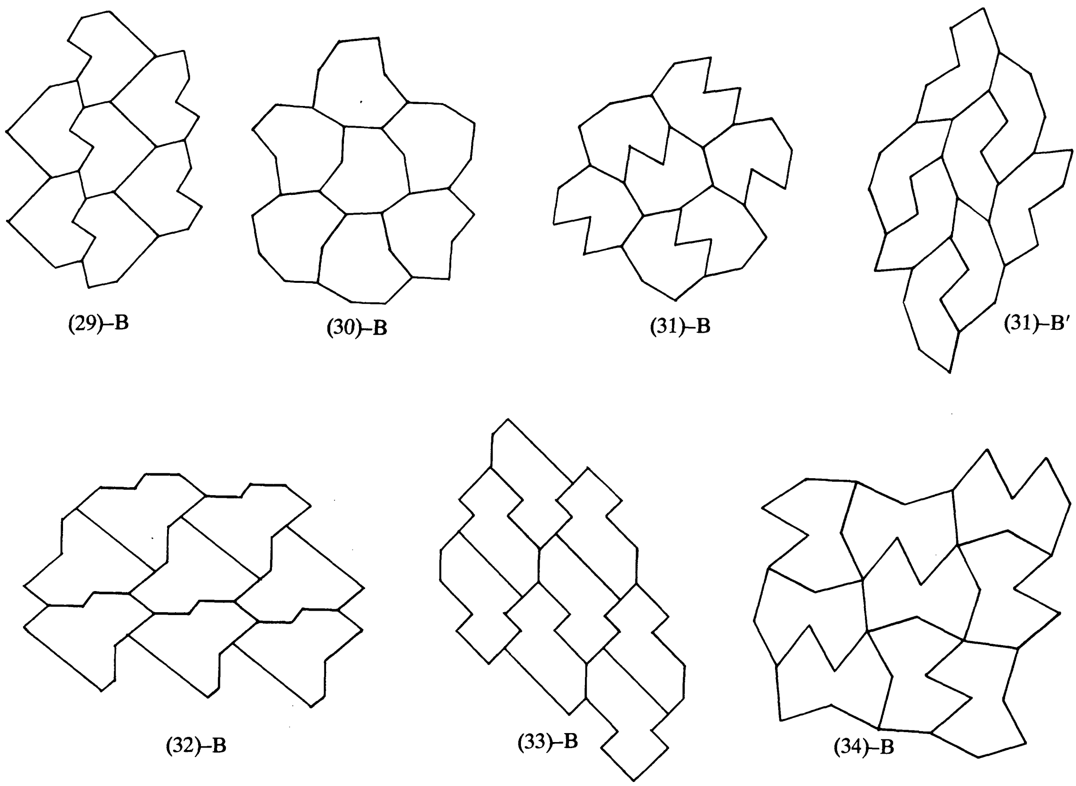
Three tilings discovered by Marjorie Rice, each containing congruent pentagons having four equal sides. The tile in (34) is type 9, and was a new addition to Kershner's list. The tile in (35) is type 1, the tile in (36) is type 2. All tilings are 2-block transitive, and for type 9, no tile-transitive tiling exists.

FIGURE 8.

FIGURE 9 shows the surprising fact that Rice's tiling (34) of type 9 pentagons has as its associated block tiling one in which each block is surrounded by just 4 vertices of the tiling. Thus tiling (34) is produced by dissecting a transitive tiling of topological quadrilaterals (Rice's tiling (36) is also obtained from 'quadrilateral' blocks). This dissection of a 'quadrilateral' tiling to produce a pentagonal tiling is most unexpected, since ordinary quadrilateral tiles (convex or non-convex), when bisected, produce either triangles or new quadrilaterals.

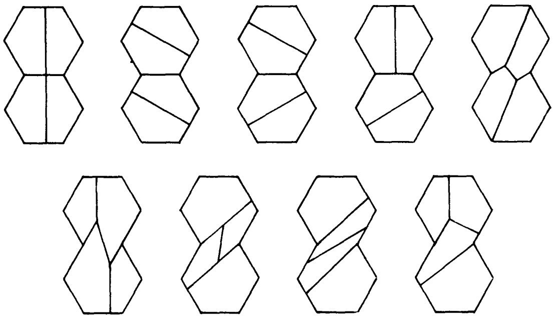
It is now easy to speculate that new types of pentagons which tile can be discovered by considering blocks of two or more congruent pentagons and determining if such blocks can tile transitively. A preliminary version of this article prompted Rice to examine a particular family of blocks, with hopes of determining new 2-block transitive pentagonal tilings. She observed that several of the 2-block transitive tilings previously discussed could also be viewed as tilings by blocks of four pentagons where these larger blocks had the outline of two hexagons stuck together. FIGURE 10 contains a schematic diagram of these "double hexagon" blocks, with the types of dissections of these blocks considered by Rice. Over sixty 2-block transitive tilings of pentagons were discovered in this way, some previously known ((25) and (26), for example), and some new. Best of all, two of the new tilings showed new types of pentagons! Just as this article was going to press, Marjorie Rice discovered yet another new tile as the result of a further search for new 2-block transitive tilings. FIGURE 10 shows the tilings associated to these new tiles (types 11, 12, and 13 on Table II).

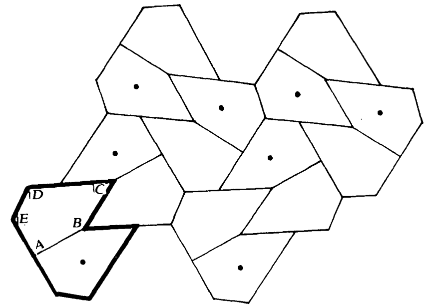
It is quite likely that still other new types of pentagons which tile can be discovered by considering dissections of transitive block tilings. The enumeration of the 81 types of isohedral tilings in **[6]** (complete with helpful diagrams) makes this task feasible. Checking possibilities will be an extremely



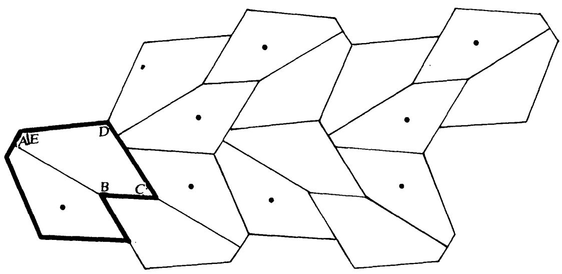
Transitive tilings by blocks outlined in tilings (29) through (34).

FIGURE 9.

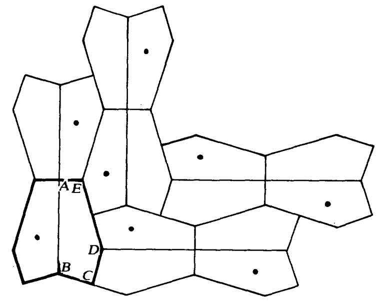




(38)



(37)



(39)

Marjorie Rice's schematic drawings of the "double hexagon" blocks of 4 pentagons. By adjusting sides and angles, many blocks of 4 congruent pentagons which tiled were found. Tilings (37) and (38) are for pentagons of types 11 and 12, respectively. Both are formed by blocks of the type shown at the lower left in the schematic diagram. Tiling (39) is for a type 13 pentagon, and is formed by blocks like those shown at the upper left in the schematic diagram.

FIGURE 10.

lengthy task, however, with a great deal of built-in repetition. This is assured by the fact that a given pentagonal tiling may have many distinct n-blocks which produce transitive block tilings and the additional fact that a single pentagonal tile may produce many distinct tilings. In order to determine if the list of types is complete, a theorem is needed to put some kind of bound on the possibilities. For the known pentagons which tile, there always exists an n-block transitive tiling for $n \leqq 3$. It is natural to hope that the following theorem is true: A pentagon tiles the plane only if there exists an n-block transitive tiling by that pentagon for $n \leqq 3$.

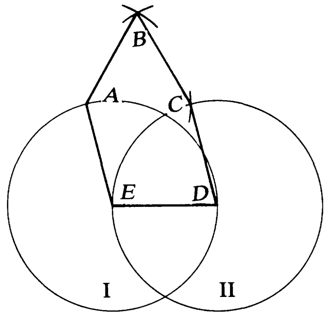
Equilateral pentagons which tile

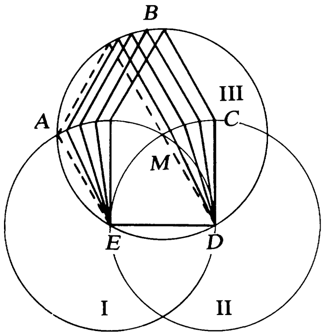
Although regular pentagons cannot tile the plane, a surprising variety of equilateral pentagons do. Of the 13 types of pentagons which tile it is obvious that types 3, 10, 11, 12, and 13 cannot be equilateral. Using construction techniques, together with familiar trigonometric relations, and the extension of these relations found in [**20**], we determined all possible equilateral pentagons of known types which tile. Types 1 and 2 provide distinct infinite families of equilateral pentagons. For types 4, 7 and 8, there is a unique equilateral pentagon of each type. Types 5, 6 and 9 cannot be equilateral. In order to investigate the equilateral case for types 7, 8 and 9, we studied the general class of equilateral pentagons $PQRST$ satisfying the condition $2P + Q = 2\pi$. FIGURE 11 contains information on this class. TABLE III contains a summary of details for all known types of equilateral pentagons which tile.

Since any two congruent equilateral pentagons will match edge to edge in any order, the possibilities for tilings are great. In attempting to determine all equilateral pentagons which tile, Marjorie Rice produced many interesting tilings. In addition, Martin Gardner's article inspired

TABLE III. Equilateral Pentagons Known to Tile. Types 1–2

Type 1. $D+E=\pi$





Construction

Choose angle E.
Construct sides $AE = ED$ by drawing circle I with center at E. Construct circle II with center D and radius DE. Find vertex C on circle II so that $DC \| AE$. Construct $CB = AB$. The pentagon $ABCDE$ has the outline of an equilateral triangle atop a rhombus. Draw circle III with radius equal to DE, and with center M, the intersection of circle I and circle II. Then for each pentagon $ABCDE$, vertex A lies on circle I, vertex C lies on circle II, and vertex B lies on circle III.

The diagram shows representatives of this type for $\frac{\pi}{2} \leqq E < \frac{2\pi}{3}$.

Angles

$$\frac{\pi}{3} < E < \frac{2\pi}{3} \qquad A = \frac{4\pi}{3} - E \qquad C = \frac{\pi}{3} + E$$

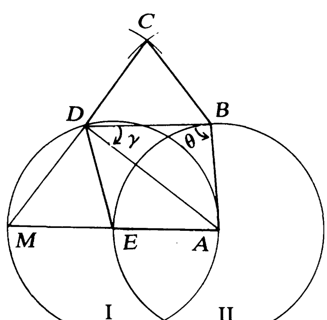
$$B = \frac{\pi}{3} \qquad D = \pi - E$$

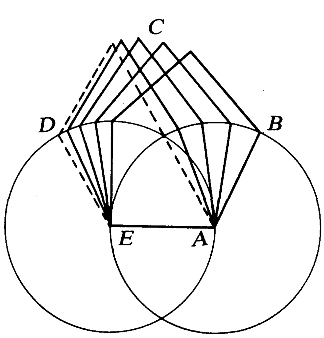
Illustrative tilings

Tile-transitive:
(4), (5), (6), (7), (10),
(11), (12), (15), (16), (24).
For $E = \frac{\pi}{2}$, (1) and (18).

Non-transitive:
(25)
For $D = 80°$, (35), (40), and (41).

Type 2. $C+E=\pi$





Construction

Choose angle E.
Construct sides $AE = ED$ by drawing circle I with center at E. Extend EA to M on circle I. Construct circle II with center A and radius AE. Find point B on circle II so that $DB = DM$; construct $\Delta DCB \simeq \Delta DEM$, (so $C = \pi - E$). Join AB to form pentagon $ABCDE$.

To obtain the description of the angles, draw segment DA, and denote $\gamma = \sphericalangle BDA$ and $\theta = \sphericalangle DBA$. Note ΔMDA is a right triangle, with $\sphericalangle DMA = E/2$ and $\sphericalangle DAM = C/2$. These facts, together with the law of cosines for ΔDBA lead to the angle relationships given.

The diagram shows representatives of this type for $\frac{\pi}{2} \leqq E < \frac{2\pi}{3}$.

Angles

$$\frac{\pi}{3} < E < \frac{2\pi}{3} \qquad \theta = \arccos\left(\frac{1+4\cos E}{4\cos E/2}\right)$$

$$C = \pi - E \qquad D = \frac{\pi}{2} + \gamma$$

$$\gamma = \arccos\left(\frac{3}{4\sin E}\right); \qquad B = E/2 + \theta$$

$$\frac{\pi}{6} < \gamma \leqq \arccos(3/4) \qquad A = \frac{3\pi}{2} - E/2 - \theta - \gamma$$

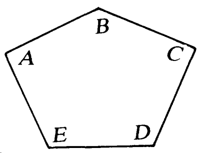
Illustrative tilings

Tile-transitive:
(8), (20), (21).
Non-transitive: (36)

Special cases:
Types 4 and 8 below.

Type 4.

$A = C = \frac{\pi}{2}$



Construction

Follow the construction of type 2 above for chosen angle $E = \pi/2$, then re-label the vertices of the pentagon replacing E by A, D by B, etc., in clockwise order. Angles can be established easily using right triangle relationships.

Angles

$$A = C = \frac{\pi}{2} \qquad B = 2\pi - 2D \sim 131°24'$$

$$D = E = \frac{\pi}{4} + \arccos\frac{\sqrt{2}}{4} \sim 114°18'$$

Illustrative tiling: (3) and tilings listed for type 2 above.

Type 5.

$A = \frac{\pi}{3},\ C = \frac{2\pi}{3}$

Impossible. The conditions $A = \pi/3$, $C = 2\pi/3$ imply that the pentagon would have to be of type 2 above. For $A = \pi/3$, the construction yields a limiting quadrilateral of the type 2 family, i.e., an equilateral "pentagon" with $E = \pi$.

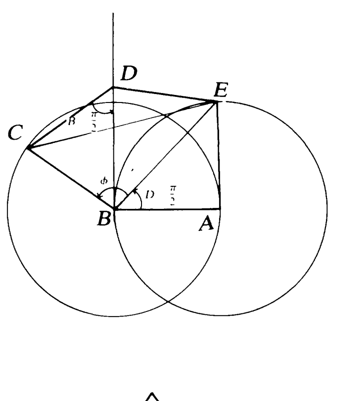
Type 6.

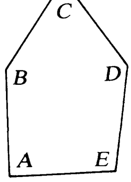
$C + E = \pi,\ A = 2C$

Impossible. The conditions $C + E = \pi$ and $A = 2C$ imply that the pentagon would have to be of type 2 above, with $\pi/3 < C < \pi/2$. Then $\pi/2 < E < 2\pi/3$ implies $-1 < \cos E < -1/2$ and $\cos E/2 > 0$, so arcos $\theta < 0$. Thus $\theta > \pi/2$, and since $\gamma > \pi/6$, we have $3C/2 = A - C/2 = \pi - \theta - \gamma < \pi/3$, a contradiction to $\pi/3 < C$.

Type 7.

$2B + C = 2\pi,\ 2D + A = 2\pi$





Construction

Let $B = P$ and $C = Q$ in FIGURE 11; then $D = R$, $E = S$, $A = T$. Adjust vertex R on line l until $2D + A = 2\pi$ (a unique solution). Let $\phi = \sphericalangle CBE$. The angle relations indicated in figure 11, together with the additional condition $2D + A = 2\pi$, give the following general conditions on angles:

$$B = \frac{\pi}{2} - \frac{A}{2} + \phi \qquad D = \pi - A/2$$

$$C = \pi + A - 2\phi \qquad E = \frac{\pi}{2} - A + \phi$$

The extended law of cosines in **[20]** gives $\sin\phi = 1 - \cos A$; $\phi = \pi - \arcsin(\sin\phi)$. In addition, the following equations can be obtained, the first from FIGURE 11, the second by combining law of cosines for ΔCDE and law of sines for ΔCBE:

$$\cos C - \cos E = 1/2$$

$$16\sin^4 D + 8\sin^2 D + 4\cos D = 5$$

To obtain the angle approximations given, we noted by construction that A was close to 89°. We then computed the angles of the pentagon as A ranged over values within 1° of 89°, and computed the error in the above equations for this range. The angles given produce an error of less than .001 in both equations.

Angles

$A \sim 89°16'$ $\quad C \sim 70°55'$ $\quad E \sim 99°54'30''$

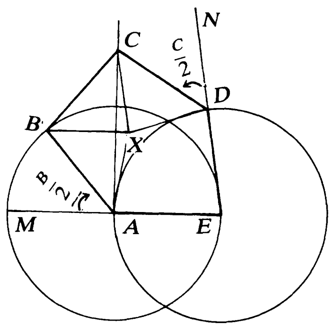
$B \sim 144°32'30''$ $\quad D \sim 135°22'$

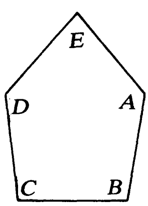
Illustrative tiling: (30)

TABLE III. Equilateral Pentagons Known to Tile. Types 8–9

Type 8.

$2A + B = 2\pi$.
$2D + C = 2\pi$





Construction

Let $A = P$ and $B = Q$ in FIGURE 11; then $C = R$, $D = S$, $E = T$. Adjust vertex R on line l until $2D + C = 2\pi$ (a unique solution). This is the unique pentagon of the family in FIGURE 11 satisfying $Q = R$, $P = S$. To see this, add construction lines to our pentagon $ABCDE$ as follows.

Extend EA to M and ED to N. Let X be the intersection of the bisectors of angles B and C; draw AX, BX, CX, DX. Then $\Delta XCD \simeq \Delta XCB \simeq \Delta XBA$ so $\sphericalangle XAB = C/2$ and $\sphericalangle XDC = B/2$. Now $\sphericalangle MAB = B/2$, and $\sphericalangle NDC = C/2$, which implies $\sphericalangle XAD = \sphericalangle XDA$. Thus $BX = CX$ and so angles B and C are equal, angles A and D are equal.

The explicit value for $\cos E$ can be obtained using the extended law of cosines in [**20**].

Angles

$E = \arccos \dfrac{\sqrt{13} - 3}{4}$
$\sim 81°18'$

$B = C = \pi - E \sim 98°42'$

$A = D = \dfrac{\pi}{2} + \dfrac{E}{2} \sim 130°39'$

$C + E = \pi$ and $B + E = \pi$ show this is also type 2.

Illustrative tilings

(31) and tilings listed for type 2 above.

Type 9.

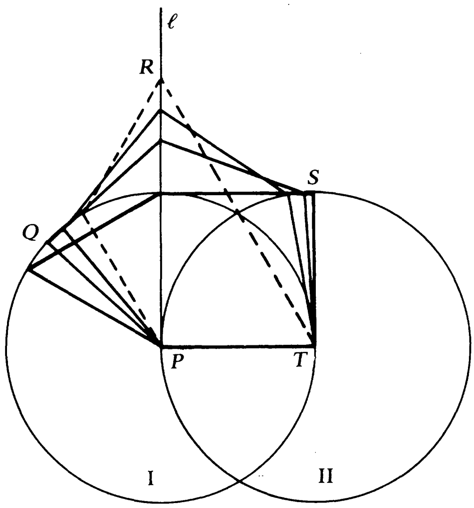
$2E + B = 2\pi$
$2D + C = 2\pi$

Impossible. The angle condition $2D + C = 2\pi$ puts it in the family in FIGURE 11, with $P = D$, $Q = C$, $E = T$. The condition $2E + B = 2\pi$ implies $E > \pi/2$, which contradicts the condition in FIGURE 11 that $E \leqq \pi/2$.

George Szekeres and Michael Hirschhorn of the University of New South Wales to conduct a week-long study group of Form 5 high school students on tilings by equilateral pentagons [**16**]. Both Rice and the Australian class discovered that a particular type 1 pentagon ($A = 140°$, $B = 60°$, $C = 160°$, $D = 80°$, $E = 100°$) could tile in curious ways (FIGURE 12). Both discovered tiling (40) composed of zigzag bands in which the pentagons can fit together in two distinct ways. The particularly beautiful tiling (41), having only rotational symmetry, was discovered by Hirschhorn. Since there are eleven ways in which the angles of this pentagon can be combined to make 360°, several other unusual tilings are also possible, including another design having only rotational symmetry. For this reason, Hirschhorn has dubbed it the "versa-tile."

Some questions

The general problem of determining all convex pentagons which can tile the plane remains unsolved. Is our list of 13 types which tile complete? We doubt it. Even though a complete solution of the problem appears to be difficult (certainly lengthy), perhaps the full answer to some special cases can be obtained. We have noted that the list of types of pentagons which can generate a tile-transitive tiling is complete (types 1 through 5). Are the only tiles capable of edge-to-edge tilings types 1, 2, 4, 5, 6, 7, 8 and 9? Is the list of equilateral pentagons which tile complete? I hope these questions will stimulate further activity on the problem.



$Q < \frac{2\pi}{3}$

$P > \frac{2\pi}{3}$

$2P + Q = 2\pi$

$\frac{\pi}{3} < T \leqq \frac{\pi}{2}$

$\frac{\pi}{3} < R < \pi$

$-1/2 + \cos Q = \cos S$

The family of equilateral pentagons for which $2P + Q = 2\pi$ can be envisioned as generated by a flexible equilateral pentagon, hinged at the vertices, with side PT held fixed, while R rides up and down line l (causing vertices Q and S to ride along arcs on circles I and II respectively). The formal construction is as follows:

> **Choose angle $P > 2\pi/3$. Construct sides $PT = PQ$ by drawing circle I with center at P. Construct line l perpendicular to PT at P. Find vertex R on line l so $QR = QP$ (then $2P + Q = 2\pi$). Draw circle II with center T, radius PT. Find vertex S on circle II so that $RS = ST$ and $PQRST$ is convex.**

The conditions on the angles listed above follow easily from this construction.

Three pentagons in this family are known to tile the plane. In addition to the unique type 7 and unique type 8 (which is also type 2) listed in Table III, there is a unique type 1 pentagon: $P = 150°$, $Q = 60°$, $R = 150°$, $S = 90°$, $T = 90°$.

FIGURE 11.

References

[1] A. D. Bradley, The Geometry of Repeating Design and Geometry of Design for High Schools, Teacher's College, Columbia University, 1933.

[2] Béla Bollobás, Filling the Plane with Congruent Convex Hexagons without Overlapping, Budapest. Tudomanyegyetem. Annales. Sectio Mathematica. 1963, 117–123.

[3] Stanley Clemens, Tessellations of Pentagons, Mathematics Teaching, No. 67, 18–19.

[4] J. A. Dunn, Tessellations with Pentagons, Mathematical Gazette, 55, Dec. 1971, 366–369. Follow-up correspondence Vol. 56, pp. 332–335.

[5] Martin Gardner, On tessellating the plane with convex polygon tiles, Scientific American, July 1975, 112–117.

[6] B. Grünbaum and G. C. Shephard, The 81 types of isohedral tilings in the plane, Mathematical Proceedings of the Cambridge Philosophical Society, 82 (1977), 177–196.

[7] ———, The 91 types of isogonal tiling in the plane, Transactions of the American Mathematical Society, to appear.

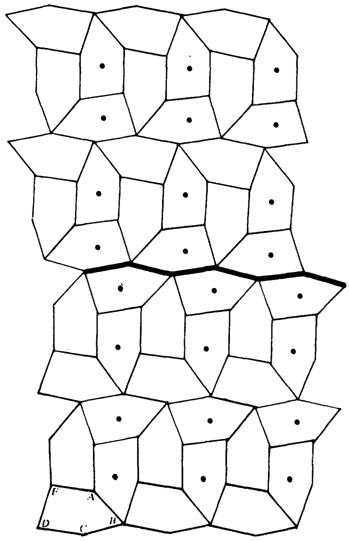
[8] ———, Isotoxal tilings, Pacific Journal of Mathematics, to appear.

[9] ———, Isohedral tilings of the plane by polygons, Commentarii Helvetici Math., to appear.

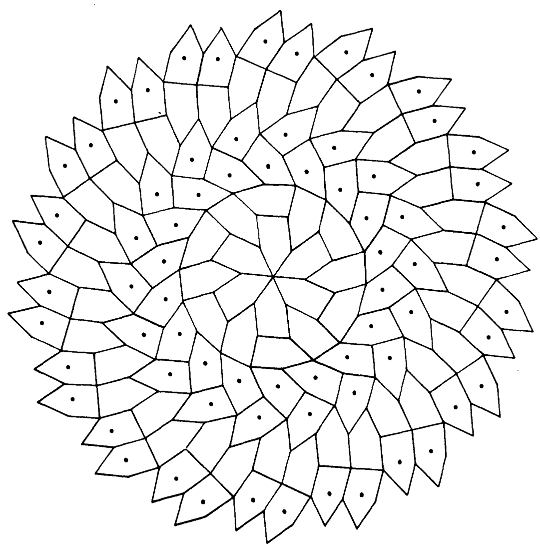
[10] F. Haag, Die regelmässigen Planteilungen, Zeitschrift für Kristallographie, 49 (1911), 360–369.

[11] ———, Die regelmässigen Planteilung und Punktsysteme, Zeitschrift für Kristallographie, 58 (1923), 478–489.

[12] ———, Die pentagonale Anordnung von sich berührenden Kreisen in der Ebene, Zeitschrift für Kristallographie, 61 (1925), 339–340.



(40)



(41)

Non-transitive tilings by the equilateral pentagon $A=140°$, $B=60°$, $C=160°$, $D=80°$, $E=100°$. Pentagons fill zig-zag horizontal strips in two distinct ways in tiling (40); these strips fit together in innumerable ways to fill the plane. Tiling (41) by Michael Hirschhorn has only 6-fold rotational symmetry.

FIGURE 12

[13] ———, Die Symmetrieverhältnisse einer regelmässigen Planteilung, Zeitschrift für mathematik und nature. unterricht, 59 (1926), 262–263.

[14] H. Heesch and O. Kienzle, Flächenschluss, Springer-Verlag, Berlin — Göttingen — Heidelberg, 1963.

[15] H. Heesch, Reguläres Parkettierungsproblem, Westdeutscher Verlag, Köln-Opladen, 1968.

[16] M. Hirschhorn, The 1976 Summer Science School. Tessellations with convex equilateral pentagons, Parabola, 13, Feb/March 1977, 2–5. More tessellations with convex equilateral pentagons, Parabola, 13, May/June 1977, 20–22.

[17] Richard James, New pentagonal tiling reported by Martin Gardner in Scientific American, December 1975, 117–118.

[18] R. B. Kershner, On Paving the Plane, American Mathematical Monthly, 75 (1968), 839–844.

[19] ———, On Paving the Plane, APL Technical Digest, 8 (1969), 4–10.

[20] ———, The Law of Sines and Cosines for Polygons, Mathematics Magazine, 44, No. 3, May 1971, 150–153.

[21] F. Laves, Ebenenteilung in Wirkungsbereiche, Zeitschrift für Kristallographie, 76 (1931), 277–284.

[22] ———, Ebenenteilung und Koordinationszahl, Zeitschrift für Kristallographie, 78 (1931), 208–241.

[23] P. A. MacMahon, New Mathematical Pastimes, Cambridge University Press, London, 1921.

[24] J. Milnor, Hilbert's problem 18: on crystallographic groups, fundamental domains, and on sphere packing, Mathematical Developments Arising from Hilbert Problems, Proceedings of Symposia in Pure Mathematics, American Mathematical Society, 1976, 441–506.

[25] Phares O'Daffer and Stanley Clemens, Geometry: An Investigative Approach, Addison-Wesley, 1976. ———, Laboratory Investigations in Geometry, Addison-Wesley, 1976.

[26] John Parker, Topics, Tessellations of Pentagons, Mathematics Teaching, 70, 1975, p. 34.

[27] Marjorie Rice, New pentagonal tilings reported to Martin Gardner and the author. (Unpublished)

[28] K. Reinhardt, Über die Zerlegung der Ebene in Polygone, Dissertation, Universität Frankfurt, 1918.

[29] D. Schattschneider, Pentagonal Tilings of Richard James's Type, (unpublished).

[30] ———, The Plane Symmetry Groups: their recognition and notation, American Mathematical Monthly, to appear.

[31] A. V. Shubnikov and V. A. Koptsik, Symmetry in Science and Art, Plenum Press, 1974.