A1. Find all ordered pairs (a, b) of positive integers for which
$$\frac{1}{a}+\frac{1}{b}=\frac{3}{2018}.$$

A2. Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n-1)\times(2^n-1)$ matrix whose (i, j) entry is
$$m_{ij}=\begin{cases}0 & \text{if } S_i \cap S_j = \emptyset;\\ 1 & \text{otherwise.}\end{cases}$$
Calculate the determinant of M.

A3. Determine the greatest possible value of $\sum_{i=1}^{10}\cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum_{i=1}^{10}\cos(x_i)=0$.

A4. Let m and n be positive integers with $\gcd(m, n) = 1$, and let
$$a_k=\left\lfloor\frac{mk}{n}\right\rfloor-\left\lfloor\frac{m(k-1)}{n}\right\rfloor$$
for $k = 1, 2, \ldots, n$. Suppose that g and h are elements in a group G and that
$$gh^{a_1}gh^{a_2}\cdots gh^{a_n}=e,$$
where e is the identity element. Show that $gh = hg$. (As usual, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.)

A5. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

A6. Suppose that A, B, C, and D are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments AB, AC, AD, BC, BD, and CD are rational numbers, then the quotient
$$\frac{\operatorname{area}(\triangle ABC)}{\operatorname{area}(\triangle ABD)}$$
is a rational number.

B1. Let $\mathcal{P}$ be the set of vectors defined by

$$\mathcal{P} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \;\middle|\; 0 \le a \le 2,\ 0 \le b \le 100, \text{ and } a, b \in \mathbb{Z} \right\}.$$

Find all $\mathbf{v} \in \mathcal{P}$ such that the set $\mathcal{P} \setminus \{\mathbf{v}\}$ obtained by omitting vector $\mathbf{v}$ from $\mathcal{P}$ can be partitioned into two sets of equal size and equal sum.

B2. Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \le 1\}$.

B3. Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

B4. Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \ge 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

B5. Let $f = (f_1, f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right)^2 > 0$$

everywhere. Prove that f is one-to-one.

B6. Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$