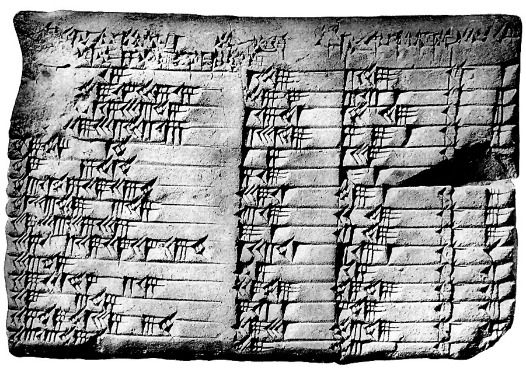
Constructing a Decimal Analogue of Plimpton 322

One of the most fascinating mathematical items is a small (approximately 13 cm $\times$ 9 cm $\times$ 2 cm) Old Babylonian clay tablet labeled Plimpton 322, which is usually dated to around 1900–1600 BCE. This tablet contains a table of numbers that are expressed in sexagesimal[1] (base 60) notation, written in the wedge-shaped cuneiform script typical of ancient Mesopotamian civilizations. The table has attracted interest from scholars because it appears to represent a list of primitive Pythagorean triples: triples of strictly positive integers (a, b, c) satisfying the equation $a^2 + b^2 = c^2$ and such that a, b, c have no common factors. Some familiar examples of such triples are $(3, 4, 5)$ and $(5, 12, 13)$, which can also be interpreted as the three sides of a right triangle.



The tablet Plimpton 322 and the historian Daniel Mansfield

The actual purpose of Plimpton 322 has long been debated, and it is also not certain how the displayed numbers were originally constructed. In this activity, you will use one of the possible algorithms for how this was done, known as the Reciprocal-Pairs algorithm, to construct an analogue of Plimpton 322 in decimal[2] (base 10) notation.

You may work in groups in order to divide up effort and check your calculations!

[1]The word *sexagesimal* comes from the Latin word *sexagesimus*, meaning ‘sixtieth.’
[2]The word *decimal* also has a Latin root: *decimus*, meaning ‘tenth.’

The numbers listed in Plimpton 322 itself, as well as those that arise when constructing them with the Reciprocal-Pairs algorithm, are all rational numbers that have only finitely many digits when expressed in base 60; that is, they have terminating sexagesimal representations. If we translate these numbers into base 10, however, some of them have infinitely many digits, or non-terminating decimal representations.[3] Before you start your construction of a decimal analogue of Plimpton 322, it is therefore important to understand which rational numbers n/d have terminating decimal representations, and also how to convert them from fractional form to decimal form. The two exercises below provide some reminders, which might also save you some computational work in the later exercises.

(1) Each of the following fractions is in lowest terms. *Without using division*, find the decimal representation of each. Do this by multiplying the numerator and the denominator by the same number in order to turn the denominator into a power of 10.

(a) $\dfrac{14}{25}$

(b) $\dfrac{33}{400} = \dfrac{33}{2^4 \cdot 5^2}$

(c) $\dfrac{431}{2^5 \cdot 5^8}$

(2) Suppose that n/d is a rational number written in lowest terms that has a terminating decimal representation. Explain why the only prime factors of the denominator d must be 2 and/or 5.

[3]For example, in base 10, we write $1/3 = 0.3333\ldots$ to represent that we need non-zero values in every one of the infinitely many places after the decimal point. However, since $1/3 = 20/60$, we can write $1/3$ in base 60 using a non-zero value in just the sixtieths place; that is, in the first place after the 'sexagesimal point,' we write the symbol (or symbols) that represent 'twenty' in the specific numeral system that we are using (e.g., cuneiform, Hindu-Arabic numerals).

Time now to get started on the construction of your decimal analogue of Plimpton 322. You will do this in a series of steps, each involving the creation of an auxiliary list or table that will help you keep track of the various numbers you will need to compute.

(3) We call a positive integer *regular for base 10* (or a *regular decimal*) if its reciprocal has a terminating decimal representation. Notice that 1 is the smallest regular decimal. By the previous exercise, the only other way for an integer to be regular for base 10 is if its prime factors are only 2 or 5. For example, $80 = 2^4 \cdot 5$ is regular for base 10, but $15 = 3 \cdot 5$ is not.

List all regular decimals from 1 to 100.

---

(4) Complete the table below as follows:

For each given value of r, find the smallest regular decimal R which is greater than r and is coprime to r (this means that R and r have no common factors other than 1). Then find the associated pair of reciprocals R/r and r/R; write these values both as fractions in lowest terms and in decimal form.[4]

r	R	R/r (fraction)	r/R (fraction)	R/r (decimal)	r/R (decimal)	row #
1						1
5						2
16						3
25						4
4						5

[4]In the 'Bonus Explorations' section of this activity, you will have a chance to explore why this table only includes five rows.

(5) For all values of r and R in your table from part (4), calculate and write down the values of:

$$L = \frac{1}{2}\left(\frac{R}{r} - \frac{r}{R}\right) \qquad \text{and} \qquad H = \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right).$$

Record these values in the table below both as fractions in lowest terms and in decimal form.

Don't worry if you don't see where these expressions for H and L are coming from yet—we'll come back to this after you have completed the full construction algorithm.

L (fraction)	L (decimal)	H (fraction)	H (decimal)	row #
				1
				2
				3
				4
				5

Before going to the next step in your construction, pause here to see if you can find any possible relationships between the values of H and L listed in the table above. Write down at least two observations or conjectures[5] that you have in the space below. Also try testing any conjecture that you come up with on all the rows in the table.

[5]Remember that Plimpton 322 is related to Pythagorean triples!

(6) For the values of L and H in your table from part (5), fill in the table below with the following values:

- the square of H, written in decimal form;
- the least numerator n_L of L;

  The 'least numerator of L' is the numerator of the reduced fraction form of L. For instance, given $L = \frac{6}{10} = \frac{3}{5}$, the least numerator is $n_L = 3$.

- the least numerator n_H of H.

Use the row ordering from the previous steps, and notice that the quantity H^2 is in decreasing order.

H^2 (in decimal form)	n_L	n_H	row #
			1
			2
			3
			4
			5

Well done! You have constructed a decimal analogue of Plimpton 322!

Pause again here to make some observations and look for patterns in your final table. Write down at least two things you notice about it. Also write down at least two questions that you have about the table or the algorithm you used to construct it.

Now, let's explore how the steps of the Reciprocal-Pairs algorithm that you used to build your decimal analogue of Plimpton 322 relate to Pythagorean Triples and to right triangles.

(7) Let's start with the question:

What does Plimpton 322 have to do with Pythagorean Triples?

Look back at each row of your table in part (5). One thing that you may have noticed is that the rational numbers L and H in each row have the same least denominator[6] Let's call this shared least denominator d.

(a) Use the table below to list the numbers (n_L, d, n_H).

n_L	d	n_H	row #
			1
			2
			3
			4
			5

(b) For each row of this table, show that the three natural numbers (n_L, d, n_H) form a *primitive Pythagorean triple*. Remember that this means checking two things:

(i) n_L, d, n_H do not have common factors and (ii) $n_L^2 + d^2 = n_H^2$.

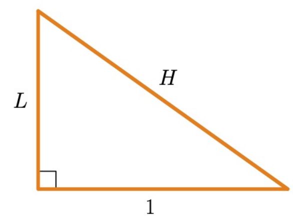
Notice that, by part (b) and the Pythagorean Theorem, the five specific values of n_L, d, n_H in your table from part (a) are the side lengths for a right triangle. In order to show that this is always the case for values of n_L, d, n_H that come from the Reciprocal-Pairs algorithm, it will be helpful to first take a closer look at an earlier step of this algorithm.

[6]Indeed, this is true every time we use the Reciprocal-Pairs algorithm. In the 'Bonus Explorations' section of this activity, you will have a chance to explore why this is always the case.

(8) Now let's move to the question:

What are the numbers L and H?

One interpretation of the rational numbers L and H that has been suggested by historians of mathematics is that they represent the lengths of the short leg and the hypotenuse of a right triangle



whose long leg measures 1. This means that the values of L and H must satisfy the Pythagorean equation,[7] with H as the hypotenuse length:

$$L^2 + 1 = H^2.$$

Recall from part (5) that the values of L and H come from the following algebraic formulas, starting with two regular decimal integers r and R:

$$L = \frac{1}{2}\left(\frac{R}{r} - \frac{r}{R}\right) \qquad \text{and} \qquad H = \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right).$$

Use algebra to verify that these expressions for L and H satisfy the equation $L^2 + 1 = H^2$.

[7]Since the long leg of the triangle measures 1 and L is its short leg, it is also necessary for L to be a positive number with $L < 1$. By inspection, you can see this is true for the entries in your table from part (5). In the 'Bonus Explorations' section of this activity, you will have a chance to explore the question of how to make sure this is always the case.

(9) Finally, we come back to the question

Can we always interpret n_H, d and n_L as the side lengths for a right triangle?

So far, you have only shown this is possible for the five triples in your table from part (7). Now that we have connected the rational numbers L and H to a right triangle via the Pythagorean Theorem, let's go back to the general case. Remember first that the whole numbers n_H, d and n_L are the least numerators and denominators of the rational numbers L and H.

Show (via the Pythagorean Theorem) that we can always interpret n_H, d and n_L as the side lengths for a right triangle. Do this by using some algebra, starting with the equation $L^2 + 1 = H^2$.

Well done! Here ends this journey through Old Babylonian mathematics!

You can use the space below to write down any closing thoughts or questions that you have about the ideas you've met along the way

Still feeling adventurous? Take a look at the 'Bonus Explorations' section of this activity.

Bonus Explorations for the Activity 'Constructing a Decimal Analogue of Plimpton 322'

(A) In part (8), you showed that the rational numbers L and H always satisfy the equation $L^2+1 = H^2$.

But as we remarked in footnote 7, in order to interpret L and H as the lengths of the short leg and the hypotenuse of a right triangle whose long leg measures 1, there is one more requirement that must hold. Namely, to make sure that L really is the short leg, it is also necessary for L to be a positive number with $L < 1$.

Of course, the value of L in turn depends on the initial values that we choose for r and R. This means that to ensure that $0 < L < 1$, we need to place some restrictions on the values of r and R.

As a bonus exploration, try your hand at proving the following fact:

The value of $L = \frac{1}{2}\left(\frac{R}{r} - \frac{r}{R}\right)$ *satisfies the inequalities* $0 < L < 1$ *if and only if the chosen values of* r *and* R *satisfy the inequality* $r < R < (1+\sqrt{2})r$.

Or, if you're not quite feeling adventurous enough to do the full proof, go back to part (4) and check that the values of r and R there satisfy the inequality $r < R < (1+\sqrt{2})r$.

(B) You may have wondered why the table you constructed in this activity contains only 5 rows, even though your list of regular decimals in part (3) is much longer. Part A helps us see why! We now know that in order to keep $0 < L < 1$, we must start with values of r and R that satisfy the inequality $r < R < (1+\sqrt{2})r$. As it turns out, the table you constructed in part (4) is complete for regular decimals r and R below 100 in the following sense:

If we pick further pairs of numbers (r, R) *from the list in part (3) with*

$$r < R < (1+\sqrt{2})r,$$

we do not obtain further values for the pair of reciprocals R/r *and* r/R.

You can check this claim out by computing a few more pairs of reciprocals R/r and r/R, starting with different choices for r and R from your list in part (3). First, check whether your selected pair satisfies the inequality $r < R < (1+\sqrt{2})r$. If it does, then verify that the reciprocal pairs r/R and R/r you get from that pair are already included in your table from part (4). To make this a more complete exploration, be sure to try at least one coprime example and one non-coprime example.

(C) One final mystery that you might have wondered about is why the rational numbers H and L produced by the Reciprocal-Pairs algorithm always share the same least denominator. This is again due to the equation from part (8): $L^2 + 1 = H^2$.

In fact, one can prove that any two rational numbers x and y satisfying the equation $x^2 + 1 = y^2$ must have the same least denominator. If you are interested in number theory, try proving this yourself as a bonus exploration. Not sure where to start? Ask your instructor for a hint!